Chapter 5: Probability Theory

Introductory Concepts:

Experiment

Sample Space

Event
Probability: 3 types

Classical Probability

Relative Frequency Probability

Subjective Probability
Law of large numbers

Expressing Probability in terms of "odds"
Concepts involving *events*. Consider the experiment of rolling a pair of fair dice.

**Event**

**Complement of an event (A̅)**

**Mutually exclusive events**

**Exhaustive (or ‘collectively exhaustive’) events**
Experiment: Roll a pair of fair dice (one red, one green)

Sample Space:

- (1,1) (1,2) (1,3) (1,4) (1,5) (1,6)
- (2,1) (2,2) (2,3) (2,4) (2,5) (2,6)
- (3,1) (3,2) (3,3) (3,4) (3,5) (3,6)
- (4,1) (4,2) (4,3) (4,4) (4,5) (4,6)
- (5,1) (5,2) (5,3) (5,4) (5,5) (5,6)
- (6,1) (6,2) (6,3) (6,4) (6,5) (6,6)

Assumption: the dice are “fair” (balanced), so each sample point has the same probability of 1/36.

Let \( A = \{ \text{Total is a 6} \} = \{(1, 5), (2, 4), (3, 3), (4, 2), (5,1)\} \).

Then \( P\{A\} = \) ________

Let \( B = \{ \text{Red die is even} \} = \{(2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\} \)

Then \( P\{B\} = \) ________

Combinations of Events:

**Union:** \( P\{A \text{ or } B\} = P\{(2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6), (1,5), (3,3), (5,1)\} \)

So \( P\{A \text{ or } B\} = \) ________

**Intersection:** \( P\{A \text{ and } B\} = P\{(2, 4), (4, 2)\} \)

So \( P\{A \text{ and } B\} = \) ________
Relating unions and intersections:

\[ P\{A \text{ or } B\} = \]

So in our example \( P\{A \text{ or } B\} = \)

[Diagram of two overlapping circles labeled A and B]
Recall the dice example:

\[ A = \{\text{total is 6}\} = \{(1, 5), (2, 4), (3, 3), (4, 2), (5,1)\} \]

\[ B = \{\text{Red (first) die is even}\} = \{(2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\} \]

\[ P\{A\} = 5/36; \; P\{B\} = 18/36; \; P\{A \text{ or } B\} = 21/36; \; P\{A \text{ and } B\} = 2/36 \]

The concept of conditional probability:

\[ P\{A \mid B\} \] denotes the conditional probability of A given B, meaning the probability that event A occurs if we know that event B has occurred.

\[ P\{A \mid B\} = \ldots \] (work in the restricted sample space!)

\[ B = \{\text{Red (first) die is even}\} = \{(2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\} \]

\[ P\{B \mid A\} = \ldots \]
Formulas for conditional Probability:

\[ P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)} \quad P(B \mid A) = \frac{P(A \text{ and } B)}{P(A)} \]
**Example Problem:** Consider the experiment of tossing a fair coin 4 times. Find the sample space. Then find: (a) \( P\{4 \text{ heads}\} \); (b) \( P\{\text{At least two heads}\} \); (c) \( P\{4 \text{ heads} \mid \text{At least two heads}\} \); and (d) \( P\{\text{At least two heads} \mid 4 \text{ heads}\} \).

Sample Space

- HHHH
- HHHT
- HHTH
- HTHH
- THHH
- HHTT
- HTHT
- HTTH
- THHT
- THTH
- TTHH
- HTTT
- THTT
- TTHT
- TTHH
- TTTT
- THTT
- TTTH
- TTTT
Let $A = \{4 \text{ Heads}\}$ and $B = \{\text{At least 2 heads}\}$

Are $A$ and $B$ mutually exclusive?

Are $A$ and $B$ collectively exhaustive?
Independence and Dependence:

Events A and B are independent if and only if ($\iff$)

\[
P\{A \mid B\} = P\{A\} \iff P\{B \mid A\} = P\{B\} \iff P\{A \text{ and } B\} = P\{A\}P\{B\}
\]

These criteria are equivalent. If one is true then they’re all true, if one is false then they’re all false.

Interpretation (1st criterion): A is independent of B if the knowledge that B has occurred doesn’t affect (doesn’t cause us to revise) the probability that A occurs.

Recall the original dice example. Are the events $A = \{\text{total is 6}\}$ and $B = \{\text{Red die is even}\}$ dependent or independent?
**Independent vs. Dependent Example:** Consider the experiment of tossing a fair coin 10 times. Let:

\[
\begin{align*}
A &= \{1^{\text{st}} \text{ toss results in a head}\} \\
B &= \{10^{\text{th}} \text{ toss results in a head}\} \\
C &= \{1^{\text{st}} 9 \text{ tosses all result in heads}\}
\end{align*}
\]

Are A and B independent? B and C? A and C?
A Note on Probabilities Involving Sequences of Events

Recall the conditional probability formula:  \[ P(B \mid A) = \frac{P(A \text{and} B)}{P(A)} \]

This can be rewritten as:  \[ P(A \text{ and } B) = P(A) \cdot P(B \mid A) \]

which you can think of as saying that the probability that A and B occur is the probability that A occurs times the probability that B occurs, knowing that A occurs.

You can extend this concept to longer sequences:

\[ P(A \text{ and } B \text{ and } C) = P(A) \cdot P(B \mid A) \cdot P(C \mid A \text{ and } B) \]

and so on. Make sure that conditional probabilities are used as you work through the sequence. You must take past information into account as you go!

If A, B, and C are independent then we don’t need to worry about conditional probabilities:

\[ P(A \text{ and } B \text{ and } C) = P(A) \cdot P(B) \cdot P(C) \]

This will prove to be a very useful concept!
Examples: Probabilities for sequences of events

Toss a fair coin four times

\[ P\{HTTH\} = \]

Toss an unbalanced coin, where \( P\{\text{Head on any toss}\} = 0.6 \), four times.

\[ P\{HTTH\} = \]

Deal four cards from a freshly shuffled deck.

\[ P\{\text{Ace, Ace, Ace, Ace}\} = \]
Example: Deal two cards from a freshly shuffled deck. What’s the probability of obtaining a blackjack?
Experiment: Select a committee of 2 from 5 Students. Three of the five are male (M1, M2, and M3), two are female (F1, F2). Assume that selections are made randomly, so that each possible committee has the same probability. Here’s the sample space:

- M1 & M2
- M1 & M3
- M1 & F1
- M1 & F2
- M2 & M3
- M2 & F1
- M2 & F2
- M3 & F1
- M3 & F2
- F1 & F2

P{Committee has 2 males} = ________
P{Committee has 1 male} = ________
P{Committee has 0 males} = ________

OR..... Draw out a probability tree. Probabilities are obtained by multiplying along the paths.
More practice on conditional probabilities:

Suppose “M1” is Bob. \( P\{\text{Bob gets on the committee}\} = \) ________

\[ P\{\text{Bob gets on the committee} \mid \text{committee has exactly one male}\} = \] ________

\[ P\{\text{Bob gets on the committee} \mid \text{committee has at least one male}\} = \] ________

Are the events \( A = \text{“Bob gets on the committee”} \) and \( B = \text{“Committee has at least one male”} \) independent or dependent? Why?
Two-Way Table Example: Marginal and Joint Probabilities

Contingency Table (Cross-tabulation) Breakdown of Tennis Camp Attendees:

<table>
<thead>
<tr>
<th>Sex</th>
<th>Beginner</th>
<th>Intermediate</th>
<th>Advanced</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Male</td>
<td>15</td>
<td>35</td>
<td>10</td>
<td>60</td>
</tr>
<tr>
<td>Female</td>
<td>5</td>
<td>45</td>
<td>30</td>
<td>80</td>
</tr>
<tr>
<td>Total</td>
<td>20</td>
<td>80</td>
<td>40</td>
<td>140</td>
</tr>
</tbody>
</table>

Joint and Marginal Probabilities for Tennis Camp Attendees:
(Divide every entry in the first table by 140 to obtain this table)

<table>
<thead>
<tr>
<th>Sex</th>
<th>Beginner</th>
<th>Intermediate</th>
<th>Advanced</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Male</td>
<td>.107</td>
<td>.250</td>
<td>.071</td>
<td>.429</td>
</tr>
<tr>
<td>Female</td>
<td>.036</td>
<td>.321</td>
<td>.214</td>
<td>.571</td>
</tr>
<tr>
<td>Total</td>
<td>.143</td>
<td>.571</td>
<td>.286</td>
<td>1.000</td>
</tr>
</tbody>
</table>

Find:

P{A randomly selected player is a beginner}

P{..... is a female}

These are marginal probabilities.

P{..... is an advanced male}

This is a joint probability.
If I choose a player at random and tell you that this player is a male, what's the probability that this player is advanced?

If I choose a player at random and tell you that this player is a beginner, what's the probability that this player is female?

Are the events "beginner" and "female" independent or dependent? Why?
Once again, consider the committee selection problem. What if the committee problem involved a larger group, and it was impractical to list the sample points? Use **counting rules**: principle of multiplication, combinations, and permutations.

**The principle of multiplication:** If the first part of an experiment has $m$ possible outcomes and the second part has $n$ possible outcomes, then there are $mn$ possible outcomes for the overall experiment (multiply!). The obvious extension holds for experiments with more parts!

Example: How many different Jeep Cherokees are available if you have 2 engine choices, 6 color choices, 2 transmission choices, and 3 driveline choices? Also, why don’t they get better mileage, given their weight (bonus question)?

**Combinations:** The number of ways in which you can select $r$ distinct objects from a pool of $n$ distinct objects is given by: $$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$ This is read as “a combination of $n$ things taken $r$ at a time.” We’re assuming that the order of selection doesn’t matter – we just care about which objects get chosen.

Example: Going back to our committee problem, the total number of possible committees is given by:
The number of committees having two males and zero females is given by:

so the probability of a two male, zero female committee is ________.

The number of committees having one male and one female is given by:

so the probability of a one male, one female committee is ________.

Finally, the number of committees having zero males, two females is given by:

so the probability is ________.

This approach will also work for larger problems!
Example Problem: How many possible committees of size four could be formed using members of this class?
**Permutations:** The number of ways in which you can select \( r \) distinct objects from a pool of \( n \) distinct objects *where a different order of selection is considered to be a different result* is given by: \( \frac{n!}{(n-r)!} \). This is read as “a permutation of \( n \) things taken \( r \) at a time.

Example: Consider a horse race with 10 entries. How many possible 1–2–3 finish combinations are there?
Bayes' Theorem and Why You Don't Need It To Solve Bayes' Theorem Problems (Text Section 5.6)

Terminology & Notation:

States of Nature: A collection of mutually exclusive, collectively exhaustive events of interest, denoted $A_1, A_2, \ldots, A_k$.

Prior Probabilities: An initial set of probabilities assigned to the states of nature, denoted $P(A_1), P(A_2), \ldots, P(A_k)$.

Posterior Probabilities: A revised (conditional) set of probabilities for the states of nature. We observe some event $B$, related to the states of nature. The probabilities $P(A_1|B), P(A_2|B), \ldots, P(A_k|B)$ represent revised state of nature probabilities that take the observation of $B$ into account. These probabilities are called posterior probabilities.

Bayes' Theorem provides the formula needed to revise prior probabilities, taking new information into account, to obtain posterior probabilities.

Assume that we know the prior probabilities for the states of nature and that we know $P(B|A_1), P(B|A_2), \ldots, P(B|A_k)$, the probabilities that $B$ occurs given each state of nature. Then, for any state of nature $A_i$ of interest, the posterior probability is calculated according to:

$$P(A_i|B) = \frac{P(A_i \text{ and } B)}{P(B)} = \frac{P(A_i \text{ and } B)}{\sum_{j=1}^{k} P(A_j \text{ and } B)} = \frac{P(A_i)P(B|A_i)}{\sum_{j=1}^{k} P(A_j)P(B|A_j)} .$$

(The final quotient is the Bayes' theorem formula.)

Note: Any problem that can be solved using Bayes' theorem can also be solved by setting up a two-way table.
Example Application:

An oil company purchases a tract of land for which there are three possible states of nature considered: Large Oil Deposit ($A_1$), Small Oil Deposit ($A_2$), or No Oil Deposit ($A_3$). Based on experience with other land parcels in the same area, they assess prior probabilities:

$$P(A_1) = 0.1; \quad P(A_2) = 0.6; \quad P(A_3) = 0.3.$$  

To learn more about the property, seismic testing is done. The firm doing the testing claims that, if a tract has a large oil deposit, the test result is "positive" 90% of the time. Small deposit tracts yield positive test results 60% of the time, and no-deposit tracts yield positive results 20% of the time. Letting $B$ represent the event of a positive test result, we have:

$$P(B \mid A_1) = 0.9; \quad P(B \mid A_2) = 0.6; \quad P(B \mid A_3) = 0.2.$$  

Now suppose that the test result is positive. What are the posterior (revised) probabilities for the three states of nature?

We’ll answer this question (and other questions) by setting up a two-way table:

<table>
<thead>
<tr>
<th>States of Nature</th>
<th>$A_1$: Large Deposit</th>
<th>$A_2$: Small Deposit</th>
<th>$A_3$: No Deposit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Positive Test Result</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Negative Test Result</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Here’s the completed table:

<table>
<thead>
<tr>
<th>State of Nature</th>
<th>A₁: Large Deposit</th>
<th>A₂: Small Deposit</th>
<th>A₃: No Deposit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Positive Test</td>
<td>90</td>
<td>360</td>
<td>60</td>
</tr>
<tr>
<td>Result</td>
<td>10</td>
<td>240</td>
<td>240</td>
</tr>
<tr>
<td>Negative Test</td>
<td>100</td>
<td>600</td>
<td>300</td>
</tr>
<tr>
<td>Result</td>
<td>100</td>
<td>600</td>
<td>300</td>
</tr>
</tbody>
</table>

Suppose that the test result is positive. Find the revised probabilities for the three states of nature. Do the probabilities change, when compared to the priors, in a logical way?
A1: Large Deposit | A2: Small Deposit | A3: No Deposit
---|---|---
Positive Test Result | 90 | 360 | 60 | 510
Negative Test Result | 10 | 240 | 240 | 490
| 100 | 600 | 300 | 1000

Suppose that the test result is negative. Find the revised probabilities for the three states of nature. Do the probabilities change, when compared to the priors, in a logical way?

<table>
<thead>
<tr>
<th>State of Nature</th>
<th>Prior Probability</th>
<th>Posterior (Revised Probability)</th>
<th>Direction of Change</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Finally, find the *unconditional* probabilities of a positive and of a negative test.
1. An experiment consists of rolling a pair of dice, one red and one green. Define the following events:

   \( A = \) the total is an eight
   \( B = \) the green die is a three
   \( C = \) the red die is an odd number

Here's the sample space (Red, Green):

\[
\begin{array}{ccccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\
(6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \\
\end{array}
\]

Find the following probabilities.

(a) \( P(A) \)  
(b) \( P(B) \)  
(c) \( P(C) \)  
(d) \( P(A \text{ and } B) \)  
(e) \( P(A \text{ and } C) \)  
(f) \( P(A \text{ or } B) \)  
(g) \( P(A \text{ or } C) \)  
(h) Suppose that someone rolls the dice behind your back and tells you that the event \( B \) has occurred (the green die is indeed a "3"). Based on this information, now what is the probability of event \( A \) (that the total is an eight)?
2. In a recent article Wilkes notes that consumer fraud – shoplifting, fraudulently cashed checks, changing prices, etc. – costs retailers over $2 billion annually. Through elaborate and expensive security systems retailers can counteract this fraud. For instance, a particular retail establishment claims that 60% of the shoplifters in this store will be detected by the store's closed-circuit TV system, 40% by the store's security officers, and 15% by both. Assuming their claims to be true, what is the probability a shoplifter is detected?
3. Each year, Harvard University receives thousands of applications for its prestigious (and, many believe, highly over-rated) MBA program. In order to give the MBA classes an international flavor, preference is given to foreign applicants. If an applicant is an American citizen, he/she has a 5% chance of being accepted. However, if the applicant is foreign, he/she has a 20% chance of being accepted. 90% of all applicants are American citizens.

(a) What is the probability that a randomly selected applicant is both foreign and rejected?
(b) What percentage of applicants are accepted?
(c) Suppose that we are told that a particular applicant was accepted. What is the probability that this applicant is an American citizen?
(d) Are the events “is an American citizen” and “is rejected” independent or dependent?
4. A large municipal bank has ordered six minicomputers to distribute among their six branches in a certain city. Unknown to the purchaser, three of the six computers are defective. Before installing the computers, an agent of the bank selects two of the six computers from the shipment, thoroughly tests them, and then classifies each as either defective or non-defective.

Find the probabilities of obtaining zero, one, and two defective items (three separate questions!) in the sample of two.
5. A manufacturer of AM/FM cassette car stereo units buys its tape drive mechanisms from two sources, A and B. Source A provides 70% of the mechanisms, source B provides 30%. Past experience shows that 3% of the mechanisms provided by Source A are defective, while 5% of those provided by Source B are defective.

Suppose that the manufacturer tests a finished unit and discovers that the tape drive is defective. What is the probability that the drive came from Source B?

Hint: Use Bayes rule, or set up a two-way table based on 1000 hypothetical tape drives.
6. A state highway department has contracted for the delivery of sand, gravel, and cement at a construction site. Because of other work commitments and labor force problems, contracting firms cannot always deliver items on the agreed delivery date. From past experience, the probabilities that sand, gravel, and cement will be delivered on the promised delivery dates by the contracting firms are 0.3, 0.6, and 0.8, respectively. Assume that the delivery or nondelivery of one material is independent of another.

(a) Find the probability that all three materials will be delivered on time.
(b) Find the probability that none of the three materials will be delivered on time.
(c) Find the probability that at least one of the materials will be delivered on time.
7. In a recent article Crawford notes that the failure rate of new products remains exceedingly high. Lack of product distinctiveness is the reason listed most often by experts seeking to understand the problem. A major publisher of books has estimated from experience that the probability that a new publication becomes a commercial success is 10\% and that the success of one publication is independent of the success of another. Suppose that during a given year the publisher introduces three new books to the newsstands of the country.

(a) What is the probability that none are a commercial success?
(b) What is the probability that at least one of the new books is a commercial success?
(c) What is the probability that no more than two of the three are commercial successes?

8. The Argentine Navy *simultaneously* launches four computer-guided torpedoes at a British ship on patrol in the Falkland Islands. Each torpedo functions independently, and has a 0.2 probability of hitting the ship. A single hit is sufficient to sink the ship.

(a) What is the probability of exactly two hits?
(b) What is the probability that the ship is sunk?

9. A mechanized inspection process has been developed for electronics parts coming off of an assembly line. The process is 90\% accurate for either good or defective parts, i.e. if a part is good, it has a 0.90 probability of being accepted and, if a part is defective, it has a 0.90 probability of being rejected. We know that the assembly line produces 80\% good parts, 20\% defective parts.

(a) What is the probability that a part is both good and accepted?
(b) What is the probability that a part is both defective and accepted?
(c) What is the probability that a part is accepted?
(d) Suppose that we buy one of the parts (which means, of course, that the part was accepted in the inspection process). What is the probability that the part is good?

10. A certain article is visually inspected successively by two inspectors. When a defective article comes through, the probability that it gets by the first inspector is 0.1. Of those that do get past the first inspector, the second inspector will miss four out of ten. What fraction of the defectives will get by both inspectors?