A decomposition-accumulation model for layered manufacturing fabrication

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1. Introduction
Layered manufacturing fabrication has been proved an effective prototyping process in assisting product development due to its advantages of reducing the product development cycle, shortening product development time, and enhancing product quality. Different layered manufacturing processes and fabricating systems have been developed and commercialized for various applications (Wohlers, 1999, Lin, 1997). Among them, Stereolithography Apparatus (SLA), the first layered manufacturing process commercialized in the industry, solidifies the liquid resin on the selected region by an ultraviolet laser. After one layer solidifies, fresh resin emerges to form a new layer continuing the processing. Thus, a three-dimensional solid object is built up layer by layer through this process (Jacobs, 1996). Laminated Object Manufacturing (LOM) cuts sections from coated paper and laminates them layer by layer to construct 3D solid objects (Warner and Hsieh, 1992). Selective Laser Sintering (SLS) produces the section layers by selectively sintering the powder, e.g. plastic, wax, ceramics, or metal particles, through a CO2 laser beam (Wu et al., 1992). In Fuse Deposition Modeling (FDM), material is melted and extruded from thin nozzles and, with the movement of controlled nozzles, the fine molten material is paved in the processing layer, fusing together with previous layers to form a 3D prototype (Comb et al., 1994). A new layered manufacturing process, Shape Deposition Manufacturing (SDM), which combines the processes of material addition and material removal through numerically controlled machining has also been recently reported (Weiss, 1997).

While much work has been done on various layered manufacturing processes, including processing planning, building orientation and optimization (Sreeram and Dutta, 1995; Frank and Fadel, 1995; Cheng et al., 1995; Xu et al., 1997), and adaptive slicing (Kulkarni and Dutta, 1995; Sabourin et al., 1996), a mathematical model to describe the process of layered manufacturing has not been found. The purpose of this study is to further develop the authors’ previous defined mathematical description for layered manufacturing processing (Lin et al., 1999) and to characterize and categorize the
 principle of layered manufacturing fabrication in terms of model decomposition and material accumulation. Section 2 describes the process of both model decomposition (layer subtraction for 3D model slicing) and material accumulation (layer addition for prototyping fabrication). In section 3, a 3D design model is descriptively represented by a set of points, and sequence functions are used to correlate the layered processing information (layer slicing and layer addition) with the selected point sets. Iso-sequence planes are then defined as the processing layers to collect points with the same processing sequence. Material is accumulated along the gradient direction of iso-sequence plane. Examples of using the proposed mathematical model to describe the layered manufacturing to process flat and no-flat surfaces and a description of layered processing error are presented in Sections 4 and 5, respectively.

2. The decomposition-accumulation process of layered manufacturing fabrication

We characterize the principle of layered manufacturing fabrication into the following two processes: model decomposition and material accumulation (Figure 1). According to this characterization, in model decomposition a CAD model or a continuous volumetric object is sliced into discrete layers and the processing paths for layer fabrication are generated based on the layer information.

In material accumulation, the discrete layers are fabricated and stacked one by one to physically rebuild the prototype. The hierarchical structure to use this decomposition-accumulation process in order to categorize current layered manufacturing processes is presented in Figure 2.

According to this hierarchy, a 3D volume is decomposed along one of the three Cartesian coordinate axes hierarchically to form a series of discrete entities (surfaces, lines and points), which are then stacked into a physical object through the material accumulation. The hierarchy in the process of model decomposition can be further divided into three levels of sub-decompositions: body decomposition, surface decomposition, and line decomposition. Body decomposition slices a 3D volumetric object into 2D section layers and it is the fundamental decomposition among the above three sub-decompositions because it exists in all types of layered manufacturing processes. After the body decomposition, some layered manufacturing processes require further decomposition in order to define the processing paths. This results in the second level surface decomposition, which divides a prescribed surface into a series of discrete lines. The third level decomposition decomposes these discrete lines into a series of discrete points.

Figure 1 Decomposition-accumulation process for layered manufacturing fabrication

![Diagram of decomposition-accumulation process for layered manufacturing fabrication](image1)

Figure 2 Hierarchy of decomposition-accumulation process

![Diagram of hierarchy of decomposition-accumulation process](image2)
We define this level as the line decomposition. Although different fabrication methods for material accumulation may result in different levels of decomposition, the concept of using model decomposition and accumulation and its sub-decompositions to describe the principle of layered manufacturing is generally applicable. For example, Laminate Object Manufacturing system requires the body decomposition, while other systems may require all three levels of decompositions.

Out of the three levels of decomposition, body decomposition will produce more error than the other two due to the limitations on the layer thickness of the accumulated material and the efficiency of the fabrication, e.g. the staircase appearance along the prototyping surface is caused from the error of the body decomposition. Usually, the interval of the layer sliced by body decomposition falls between 0.1mm to 0.25mm and the resolution from 4/mm to 10/mm. This is much lower than that in surface decomposition and line decomposition, in which the resolution densities are limited only by the size of powdered particulate and the movement resolution ratio of the control system. For most layered manufacturing systems, the dimension of powdered particulate and the resolution ratio of the movement could be in the range of 0.01mm, and their resolutions could be greater than 100/mm.

3. A mathematics model of layered manufacturing fabrication

Mathematically, a 3D object could be represented as a collection of point sets $P$ within the bounded surface. We denote a 3D volumetric object as $V$, and its entire exterior closed surface $S$. $V$ is the combination of all points within $S$, denoted by $P_a$ and on $S$, denoted by $P_s$:

$$V = P_a \cup P_s$$

(1)

$S$ is assumed to be a continuous, closed and non-self-intersecting surface. For a partitioned surface, $S$ is the union of the partitioned areas $S_i$:

$$S = S_1 \cup S_2 \cup S_3 \cup \ldots \cup S_i$$

(2)

where, $i = 1, 2, 3, \ldots m$, represents the number of the partitioned areas. For example, the stl format, a de facto standard graphical interface for layered manufacturing, represents the part’s surface by a set of triangular facets. In this case, $S_i$ corresponds to the triangular facet and $S$ is the union of all the facets that constitute the entire surface of the 3D volumetric object.

In a layered manufacturing process, layer is a fabricating unit and can be considered as the material group collectively defined by the geometrical points and sequence. The sequence is the order of the layer associated with material accumulation and defined onto the 3D volumetric object. Once the process method of layered manufacturing is determined, the sequence will be assigned to the points within the 3D volumetric object to indicate to which layer they should belong. This use of sequencing to process layers can be considered a main feature of layered manufacturing fabrication.

According to the ordered sequences, the 3D volumetric object is sliced into layers during the model decomposition, and stacked back with materials during the material accumulation. The decomposition and accumulation will follow the same sequence, otherwise, the prototyping geometry may be altered from the original design.

The sequence is a number dispersal in 3-dimensional space $\mathcal{V}$ in which the layered manufacturing process is realized. We apply a scalar function $\phi(x, y, z)$ in the space $\mathcal{V}$ to associate the sequence with the point coordinates. With the introduction of $\phi(x, y, z)$, each point in space $\mathcal{V}$ not only possesses its coordinates, but is also mapped to a sequence space, which gives the order of the decomposition and material accumulation. In other words, the sequence function uses each point’s coordinates to determine the layer to which the point belongs and the order in which it will be processed by material accumulation.

Since sequence function $\phi(x, y, z)$ normally maps to a continuous real number set, we introduce a finite discontinuous real number set $C$ to indicate the 3D volumetric object $V$ being decomposed into discrete layers:

$$C = \{c_i; a \leq c_1 < c_2 < \ldots < c_{i-1} < c_i < c_{i+1} < \ldots < c_n < c_{n+1} \leq b, i = 1, 2, \ldots, n\}$$

(3)

where subscript $i$ represents the number of the $i$th decomposition in total $n$ decomposition, and constants $a$ and $b$
represent the values of lowermost and uppermost sequence for a particular object $V$. Based on the sequence function $\phi(x, y, z)$ and finite discontinuous real number set $C_i$, the points in space $V$ can be divided into several groups $P_i$ ($i = 1, 2, \ldots, n$):

$$
P_i = \{ (x, y, z); c_i \leq \phi(x, y, z) < c_{i+1} \},$$

$$
P_n = \{ (x, y, z); c_n \leq \phi(x, y, z) \leq c_{n+1} \}$$

(4)

where $P_i$ is the set of points whose sequences are between $c_i$ and $c_{i+1}$. The points in $P_i$ may exist inside or outside the object $V$, but only the points that belong to object $V$ will be used to rebuild the object. Therefore, the $i$th layer $v_i$ is defined as the intersection of $V$ and $P_i$;

$$
v_i = V \cap P_i$$

(5)

and

$$v_i \cap v_j = \emptyset, \ i \neq j, i, j = 1, 2, \ldots, n$$

(6)

$$V = v_1 \cup v_2 \cup \ldots \cup v_n$$

(7)

Figure 3 schematically describes the definition of 3D volumetric object $V$, its boundary $S$, and the sequence function $\phi(x, y, z)$. In this description, the sub-volume $v_i$ at the $i$th layer consists of all points whose sequences fall between $c_i$ and $c_{i+1}$ in the $V$.

If we assume that $\phi(x, y, z)$ is equal to the ordered number $c_0$ as shown in Figure 4, we thus define a series of planes in $V$:

$$\phi(x, y, z) = c_i, \ i = 1, 2, \ldots, n$$

(8)

We call those planes iso-sequence planes because the points $P_i$ on each of those planes have the same sequence number. In the model decomposition process, the iso-sequence planes intersect and slice the designed 3D object. In the material accumulation process, the iso-sequence planes form the processing layers on which material will be added. For example, in SLA layered manufacturing, the iso-sequence plane is the surface of the resin vat on which the layer will be solidified from liquid resin. In LOM layered manufacturing, the iso-sequence plane is the top surface layer on which a new layer of coated paper will be attached. In SLS layered manufacturing, the iso-sequence plane is the plane on which the fresh powder is spread and sintered.

The gradient of the sequence function can be defined as:

$$d(x, y, z) = \nabla \phi = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k$$

$$k = X(x, y, z)i + Y(x, y, z)j + Z(x, y, z)k$$

(9)

where $i, j$ and $k$ are the unit vectors in $V$ along the $X$, $Y$ and $Z$ axis respectively, and $d(x, y, z)$ is a spatial vector representing the gradient of the sequence function. It also represents the processing orientation in model decomposition and material accumulation, i.e. along the direction of $d(x, y, z)$, we disperse the 3D CAD model in the model decomposition process and add materials in the material accumulation process.

We can also express sequence function $\phi(x, y, z)$ by $d(x, y, z)$ from Equation (9):

$$\phi(x, y, z) = \phi(x_0, y_0, z_0) + \int_{x_0}^{x} X(x, y, z_0)dx + \int_{y_0}^{y} Y(x, y, z_0)dy + \int_{z_0}^{z} Z(x, y, z)dz$$

(10)

if:

$$\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial z} = \frac{\partial Z}{\partial x} = \frac{\partial X}{\partial z} = \frac{\partial Y}{\partial x}$$

(11)

where $(x_0, y_0, z_0)$ is the initial point and $\phi(x_0, y_0, z_0)$ is the initial sequence number.
If we define the module \( |d(x, y, z)| \) equal to the reciprocal of the thickness of two adjacent iso-sequence planes, \( h \):

\[
|d(x, y, z)| = \frac{1}{h}
\]

(12)

then \( |d(x, y, z)| \) can be used to represent the layer density, indicating the number of layers which can be sliced and fabricated within a given thickness unit.

4. Application in the layered manufacturing processing

In current layered manufacturing processes, the fabricating orientation is usually limited in one direction (vertical direction) and the thickness of the added material at each layer is uniform. Assume that the fabricating orientation is along Z-axis and the processing layer is parallel to the XY plane with a uniform thickness \( h \) for each layer, as shown in Figure 5, the gradient of the sequence function, \( d(x, y, z) \) can be defined as:

\[
d(x, y, z) = \frac{1}{h} k
\]

(13)

From Equation (10), we can derive the sequence function \( \varphi(x, y, z) \) as:

\[
\varphi(x, y, z) = \varphi(x_0, y_0, z_0) + \int_{z_0}^{z} \frac{1}{h} dz = \varphi(x_0, y_0, z_0) + \frac{1}{h}(z - z_0)
\]

(14)

where, \( \varphi(x_0, y_0, z_0) \) represents the initial sequence number at the position \( (x_0, y_0, z_0) \). Without loss of generality, we can assume the value of \( \varphi(x_0, y_0, z_0) \) and \( z_0 \) as zero. Thus, the sequence function \( \varphi(x, y, z) \) can be expressed as:

\[
\varphi(x, y, z) = \frac{z}{h}
\]

(15)

In this case, sequence number \( c_i \) is defined as a set of integers, and \( z = ih \) represents a series of flat planes along the Z direction. These flat planes are the processing layers used in the layered manufacturing processes. From Z coordinates we can identify to which layer a given point \( P_i(x, y, z) \) should belong.

The advantage of using the proposed mathematical model is that it can also express non-planar processing surfaces. For example, for a spherical object as shown in Figure 6, we can define the sequence function and iso-sequence planes as a series of concentric spherical surfaces and use them and layered manufacturing to prototype a spherical object. In doing so, we assume that the material accumulation is conducted in the radial direction, the gradient of the sequence function can be represented by:

\[
d = \text{grad} \varphi = \frac{1}{h \sqrt{x^2 + y^2 + z^2}} (xi + yj + zk)
\]

(17)

Let us assume that material thickness \( h \) is uniform across the radial direction for each processing layer. This defines the module of \( d(x, y, z) \) to be equal to \( 1/h \). If we set \( \varphi(x_0, y_0, z_0) \) equal to 0, and \( (x_0, y_0, z_0) \) equal to \((0,0,0)\), based on Equations (9) and (10), the sequence function \( \varphi(x, y, z) \) can be derived as:

Figure 5 Application to current layered manufacturing process

Figure 6 Concentric spherical iso-sequence planes
Figure 7 Layer process error between the sequence layer and processing layer

\[ \varphi(x, y, z) = \int_0^h \frac{1}{h \sqrt{x^2 + y^2}} \, dx + \int_0^h \frac{1}{h \sqrt{x^2 + y^2}} \, dy + \int_0^h \frac{1}{h \sqrt{x^2 + y^2 + z^2}} \, dz = \frac{1}{h} \sqrt{x^2 + y^2 + z^2} \]  

(18)

the iso-sequence planes can then be defined as:

\[ \varphi(x, y, z) = \frac{1}{h} \sqrt{x^2 + y^2 + z^2} = \]

\[ 1 \Rightarrow \sqrt{x^2 + y^2 + z^2} = h; \]

\[ \varphi(x, y, z) = 2 \Rightarrow \sqrt{x^2 + y^2 + z^2} = 2h; \]

\[ \varphi(x, y, z) = i \Rightarrow \sqrt{x^2 + y^2 + z^2} = i \cdot h \]  

(19)

which are a set of concentric spheres defined by \( \sqrt{x^2 + y^2 + z^2} = i \cdot h \) with the interval \( h \), as displayed in Figure 6.

Non-planar layered manufacturing could reduce the effect of the staircase and the layered processing error.

5. Error definition for layered manufacturing process

It is difficult for current layered manufacturing processes to produce the exact boundary of \( S \) as shown in Figure 3. The layer boundaries are usually straight with their edges parallel to the fabrication orientation. Therefore the actual boundary of the 3D prototyping part is staircase-like, and the geometry of the processing layer \( v_i \) is not equal to the real geometry defined as the sequence layer \( v_i \), as shown in Figure 7. The difference is called layered process error. We use \( e_i \) to express the layered process error between the sequence layer \( v_i \) and the processing layer \( v_i \). Based on Figure 7, we can define:

\[ e_i(v_i - v_i) \cup (v_i - v_i) \]  

(20)

The layered process error \( e_i \) is related to the oblique boundary of layer \( v_i \) and the orientation during the process of the material accumulation. For an \( n \)-layered 3D prototyping object, the total layered process error can then be expressed as:

\[ E = e_1 \cup e_2 \cup \ldots \cup e_n \]  

(21)

Since the layer processed by layered manufacturing is usually thin, we can approximate the boundary of the layer \( v_i \) as flat surfaces, as shown in Figure 8, with a unit normal vector \( n \) as its orientation. To the section shown in Figure 8, the differential representation of layered process error \( de_i \) is the area of triangle \( \Delta ABC \), which is equal to:

\[ de_i = S_{\Delta ABC} = \frac{h_i^2 \cos \alpha}{2 \sin \alpha} \]  

(22)

where, \( de_i \) represents the differential representation of layered process error, \( S_{\Delta ABC} \) indicates the area of \( \Delta ABC \), and \( \alpha \) is the angle between the local gradient of sequence function \( d(x, y, z) \) and the normal \( n \). From Figure 8, we have:

\[ \cos \alpha = \frac{|d(x, y, z) \cdot n|}{|d(x, y, z)||n|}, \]

\[ \sin \alpha = \frac{|d(x, y, z) \times n|}{|d(x, y, z)||n|} \]  

(23)

Substitute equation (23) into (22), yield:

\[ de_i = \frac{h_i^2}{2} \frac{|d(x, y, z) \cdot n|}{|d(x, y, z) \times n|} \]  

(24)

Note that the Equation (24) is always positive and includes both errors of deficient filling and over filling. Integrate Equation (24) along the \( i \)th layer’s contour, the layered process error can then be calculated as:
In which, \( dl \) is the differential length along the layer contour, and \( d(x, y, z) \) and \( n(x, y, z) \) may vary along the contour.

Equation (25) is a general expression for layered process error within a given layer. If we add all layered process errors together, the total layered manufacturing process error can be defined as:

\[
E = \sum_{i} e_i = \sum_{i} \int_{A} \frac{h_j^2}{2} \left| \frac{d(x, y, z) \cdot n}{d(x, y, z) \times n} \right| dl
\]  

(25)

The integration of Equation (26) is quite difficult because the differential length \( dl \) is based on the layer contours defined by the CAD model. For simplicity, we use the tessellated facets defined by stl format. When a CAD model is converted into stl format, the part surfaces are tessellated by a set of flat triangular facets and the layer contour is approximated by a set of linear segments, as shown in Figure 9.

Based on the stl tessellated contour and the assumption that \( d(x, y, z) \) is uniform within one layer, denoted as \( d_i \), and \( n \) is also uniform within one linear segment, denoted as \( n_j \), the layered process error from Equation (25) can be represented by:

\[
e_i = \sum_{j} \frac{h_j^2}{2} \left| \frac{d_i \cdot n_j}{d_i \times n_j} \right| l_{ij}
\]

(27)

where, \( l_{ij} \) presents the line segment of the layer contour and can be determined by intersecting the \( j \)th triangular facets and \( i \)th iso-sequence plane \( \phi(x, y, z) = c_i \).

Substituting Equation (27) into (26), we obtain the total layered process error for a given 3D object in a given layered manufacturing process:

\[
E = \sum_{i} e_i = \sum_{i} \sum_{j} \frac{h_j^2}{2} \left| \frac{d_i \cdot n_j}{d_i \times n_j} \right| l_{ij}
\]

(28)

Equation (28) could be used to predict the processing error for most conventional layered manufacturing processes with a uniform layer thickness and fabrication direction. In other words, one can apply the developed model and the error prediction to calculate the processing error for a given CAD design before prototyping fabrication. The predicted processing error can also be compared with different layered manufacturing processes and process parameters. Since the error is expressed in terms of function of \( d(x, y, z) \) or \( n(x, y, z) \), one may also obtain the optimal \( d(x, y, z) \) or \( n(x, y, z) \) with minimum layered process error by adjusting the part fabricating orientation.

6. Conclusion

A mathematical model to describe the principle of layered manufacturing fabrication is presented in this study. The model is based on the concept of model decomposition: to decompose a 3D model into a series of sequence layers, and material accumulation: to accumulate material and stack processing layer into a physical prototype. A mathematics function \( \phi(x, y, z) \) is introduced to define the sequence of processing layers and to correlate the iso-sequence planes, which consist of the point sets with the same sequence for material accumulation. The presented model can be applied for both planar layered manufacturing process and non-planar layered manufacturing process. Non-planar layered manufacturing could reduce both the effect of the staircase and the layered process error. Examples of using the derived mathematical formulation to describe...
the current layered manufacturing processing are presented.

Processing error caused by layered manufacturing is defined and formulated. This error formulation can be used to predict processing error before actual prototyping. The predicted processing error can also be compared with different layered manufacturing processes and process parameters. The optimal part fabricating orientation $d(x, y, z)$ or $\phi(x, y, z)$ can also be derived through minimization of layered process error.

References


