MEM 351 – Dynamic Systems Lab

Representations: Transfer Functions
Poles and Zeros
Damped Compound Pendulum Equations of Motion

\[ \ddot{\theta} + \frac{c}{J} \dot{\theta} + \frac{m_L g d}{J} \theta = 0 \]

**Linearized 2\textsuperscript{nd} order differential equation assumes small angles**

- \( L \)  Bar length [m]
- \( d \)  Pivot to CG distance [m]
- \( m_L \)  Mass of pendulum [kg]
- \( J \)  Moment of Inertia \([kg \cdot m^2]\)]
- \( C \)  Viscous damping coefficient \([\frac{Nms}{rad}]\)

General 2\textsuperscript{nd} order form

\[ \ddot{\theta} + 2\zeta \omega_n \dot{\theta} + \omega_n^2 \theta = 0 \]
Tedious Math: Time domain differential equation

2nd order damped system

\[ \ddot{\theta} + 2\zeta \omega_n \dot{\theta} + \omega_n^2 \theta = 0 \]

Yields complex roots

\[ s_{1,2} = -\zeta \omega_n \pm j \cdot \omega_n \sqrt{1-\zeta^2} \]

Time domain solution

\[ \theta_c(t) = e^{-\zeta \omega_n t}\left\{A_1 \cos\left(\omega_n t \sqrt{1-\zeta^2}\right) + A_2 \sin\left(\omega_n t \sqrt{1-\zeta^2}\right)\right\} \quad (1) \]

Small real root will yield long settling times

Can be shown:

- Time constant \[ T_c = \omega_n \zeta \]
- 2% settling time \[ t_s = \frac{4}{T_c} \quad (2) \]
Easier Math I: Laplace Domain

- Voltage $V(s)$ applied to motor
- Propeller spins, creating lift force $F(s)$
- Lift on lever arm $r$ creates torque $T(s)$
- Pendulum rotates angle $\Theta(s)$

![Diagram of compound pendulum with motor-propeller](image)

Motorized Propeller

$V(s)$ [Volts] $\rightarrow$ $K_m$ [Nm/Volts] $\rightarrow$ $T(s)$ Torque [Nm] $\rightarrow$ $\frac{1/J}{s^2 + \frac{c}{J} s + \frac{m_L g d}{J}}$ [rad/sec] $\rightarrow$ $\Theta(s) = \theta$

- $F = Fr$
- $T = Fr$
- $c \frac{d\theta}{dt}$
- $mg$
- Compound Pendulum
Calculating Constants

Theory: can calculate lift force if have propeller pitch and radius dimensions, air density and motor angular velocity.

Experimentally, apply known voltage $V$ and pendulum will eventually reach **steady-state**. Recall

$$J\ddot{\theta} + c\dot{\theta} + m_Lgd\sin\theta = T$$

At steady-state angular acceleration and velocity are zero. The torque at this known voltage is calculated by:

$$T\big|_{ss} = m_Lgd\sin\theta_{ss}$$

And hence

$$K_m = \frac{T\big|_{ss}}{V}$$
Open-Loop Transfer Function

\[ \Theta(s) = \frac{K_m / J}{s^2 + \frac{c}{J} s + \frac{m_L g d}{J}} = G_{ol}(s) \]

Given

\[
\begin{align*}
K_m &= 0.017 \text{ Nm/V} \\
 d &= 0.023 \text{ m} \\
 J &= 0.0090 \text{ kgm}^2 \\
 m_L &= 0.43 \text{ kg} \\
 c &= 0.00035 \text{ Nms/rad}
\end{align*}
\]

\[
\frac{\Theta(s)}{V(s)} = \frac{1.89}{s^2 + 0.039s + 10.77} = G_{ol}(s)
\]

Laplace domain OL Transfer function
OLTF Simulations

Simulink
Simulation reveals long settling time. This is consistent with the low viscous damping coefficient. Poles of the characteristic equation reveal the large oscillations. Recall from (1)

\[
\frac{\Theta(s)}{V(s)} = \frac{1.89}{s^2 + 0.039s + 10.77} = G_{ol}(s)
\]

Roots of the denominator (i.e. the poles) are:

\[
s_1 = -0.0019 + j3.28 \\
s_2 = -0.0019 - j3.28
\]

Small real root will yield long settling times

In other words, the system is bordering on the margin of stability. Recall Equations (1) and (2).
Control Designer’s Goal:

Create compensators that yield desired damping and rise time.

In other words, place poles where one wants them
Calculated the following:

\[ \omega_n = 2.50 \text{ rad/s} \]
\[ \zeta = 0.0059 \]

\[ \ddot{\theta} + 2\zeta \omega_n \dot{\theta} + \omega_n^2 \theta = 0 \]

\[ s^2 + 0.0295s + 6.25 = 0 \]

\[ s_{1,2} = -0.01475 \pm 2.5j \quad \text{poles} \]
\[ s_1 = -0.01475 + j 2.50 \]

\[ \zeta = \sin \theta \]

\[ \omega_n \sqrt{1 - \zeta^2} \]

\[ s = -0.01475 - j 2.50 \]

\[ \theta = \arctan \frac{0.01475}{2.50} = 0.0059 \]

\[ \zeta = \sin \theta = \sin 0.0059 = 0.0059 \]

\[ \omega_n = \sqrt{a^2 + b^2} = \sqrt{0.01475^2 + 2.50^2} = 2.50 \]

Matches experimental data!
Where are we going with this?

It’s called the characteristic equation because it connotes system properties.

Poles are the roots of the characteristic equation. As such, they describe stability through $\omega_n$ and $\zeta$.

Question: can we alter the locations of the poles? If we can, then we change the characteristic of the system…

Answer: This is exactly what the control engineer does. One popular method is called “pole placement” control.