The Neo-classical Growth Model
(also known as the Solow Growth Model)

There are many factors that influence the growth rate of economies. Among them are the quantity of physical capital \((K)\), the level of human capital \((H)\), the availability of natural resources \((R)\), and the efficiency or productivity with which we use our inputs \((A)\) typically referred to as technology. We can represent the output of the economy (the GDP) in a production function:

\[
Y = AF(K, H, L, R)
\]  

(1)

which captures the idea of using all our inputs into production, including labor \((L)\) through the function \(F(\cdot)\) which is modified by the productivity \((A)\) or efficiency with which we combine those inputs to produce output, \(Y\). As written above, \(F(K, H, L, R)\) is a general function where the specifics have not been defined. Production functions like this are commonly constructed from data and vary in their complexity.

A simplified and specific version of the production function takes the following form:

\[
Y = AK^{1/3}L^{2/3}
\]  

(2)

where we abstract from human capital and natural resources to focus on the two key inputs to production: capital and labor. The production function exhibits a number of properties we observe in the real world. The values of the exponents \((1/3\) on \(K\) and \(2/3\) on \(L\)) reflect the income method of accounting for GDP. In the data approximately \(1/3\) of income goes to capital and \(2/3\) to labor. In addition, if we graph how output changes as the level of physical capital increases, the graph is:

![Figure 1: GDP and Capital](image-url)
Figure 1 shows that, holding $A$ and $L$ constant, as the level of physical capital increases, the level of output also rises. However, the bowed down (concave) shape of the function also means that there are diminishing marginal returns to capital. As capital increases the amount of output rises but at a diminishing rate. Figure 2 shows how the marginal product of capital changes with the level of capital. As the level of capital increases, the additional amount of output from an additional unit of capital decreases. If you construct graphs for labor against output and the marginal product of labor, they will have the same shapes as in Figures 1 and 2. The graphs would show that output increases with labor but at a decreasing rate.

When discussing growth rates we are frequently more concerned with how GDP per capita (or average income per person) changes over time as this concept is our basic measure of standard of living. Equation (2) is the production function for the entire economy and upper case $Y$ represents total output (total GDP). However, we can transform the above production function into a per-worker production function. Define lower case $y$ as output per worker (In general, upper case letters are used for aggregate economy-wide variables and lower case letters are used for per capita variables). From equation (2), divide both sides by the size of the labor force, $L$.

$$y = \frac{Y}{L} = \frac{AK^{1/3}L^{2/3}}{L}$$

On the right-hand side, the $L$ in the numerator and denominator can be combined and we can rewrite the equation as:

$$y = A\left(\frac{K}{L}\right)^{1/3} = Ak^{1/3}$$
or simply:

$$y = Ak^{1/3}$$  \hspace{1cm} (3)

The per-worker production function simply states that income per worker depends on productivity, $A$, and $k = \frac{K}{L}$ which is the capital-labor ratio, i.e. the amount of physical capital available per worker. Note that because $k$ has the same exponent as $K$ in equation (2), if we graph $y$ against $k$ it will look just like Figure 1 and if we graph the marginal product of the capital-labor ratio against $k$ it will look like Figure 2. That is, income per worker rises with the capital-labor ratio but at a decreasing rate because there are diminishing returns to the capital-labor ratio.

**THOUGHT EXERCISE 1:** Suppose that the aggregate production function takes the form $Y = AK^{1/2}L^{1/2}$. What is the per worker production function?

**THOUGHT EXERCISE 2:** Suppose that we include human capital ($H$) in the aggregate production function and it takes the form $Y = AK^{1/3}L^{1/3}H^{1/3}$. What is the per worker production function? Let $h = H/L$ represent the average level of human capital per worker. How does the average level of human capital affect GDP per capita? Explain this relationship using graphs.

**SIDE NOTE:** Note that $y$ here represents output per worker. We frequently talk about output per person (GDP per capita). If we assume that the level of population is equal to the size of the labor force then $y$ represents GDP per capita. Now, we do know that only a fraction of the population is in the workforce, but over a long period of time that fraction is roughly constant. Call that fraction $\rho$ for participation rate and let $P$ be population such that $L = \rho P$. Then

$$y = \frac{Y}{L} = \frac{Y}{\rho P} \text{ or } \frac{Y}{P} = \frac{Y}{L/\rho} = \rho \frac{Y}{L} = \rho y$$

and thus, income per capita is a fraction of income per worker. Provided $\rho$ is roughly constant over a long period of time, any change in income per worker will be reflected directly in income per capita. For simplicity, we assume that $\rho = 1$ which makes interpretation easier, but does not change any results or intuition that we derive.

Now, to understand how income per capita changes, we need to understand what changes the capital-labor ratio, $k$, and how the capital stock, $K$, changes.
Investment, $I$, is used for two broad purposes: 1) to introduce new capital equipment; and 2) to replace and repair existing capital equipment. Thus, we can express this as:

$$I = \Delta K + \delta K$$

(4)

where $\Delta K$ represents the change in the capital stock and $\delta K$ is the depreciation of the capital stock. $\delta$ is the rate of depreciation - that is the fraction of the value of machinery, equipment, etc. that is lost over a period of time. For example, computers depreciate at a rate of roughly 25-30% per year. Other equipment loses value through being used, worn out, or becoming obsolete and generally less productive. Thus, some portion of gross investment, $I$, goes towards simply replacing or fixing the existing capital we already have and the remainder, $\Delta K$, goes towards increasing the capital stock. Thus, $\Delta K$ is referred to as net investment, our investment level after netting out the fraction spent on simply replacing worn out equipment.

Putting this investment equation into per capita terms and to relate it to our capital-labor ratio, $k$, let $i$ represent gross investment per capita. Then we have:

$$i = \Delta k + (n + \delta) k$$

(5)

which represents much the same ideas as in equation (4). Investment per capita is used for replacing the worn-out depreciated portion of the capital-labor ratio $\delta k$ and increasing the level of the capital-labor ratio $\Delta k$. The only addition here is $n$ which represents the population (or worker) growth rate. If we keep the level of capital $K$ constant, but the size of the population, $L$, increases the capital-labor ratio falls, since $k = \frac{K}{L}$. Thus, in order to maintain the same level of the capital-labor ratio we need to invest just to keep up with increases in the population growth rate.

Referring back to equation (3):

$$y = Ak^{1/3}$$

to understand how $y$, GDP per capita changes, we want to understand how the capital-labor ratio changes. Rearrange equation (5):

$$\Delta k = i - (n + \delta) k$$

(6)

which shows that net investment in $k$ is equal to the difference between the gross investment per capita and the total depreciation of the capital-labor ratio (which comes from the growth rate of the population and the wearing out of machinery). For reasons we will examine later in the course when we cover savings and investment, note that investment by firms in physical capital reflects a demand. The supply of funds for that investment comes from the savings of the economy. Thus, savings equals investment in the market where supply meets demand. Let $I$ be the economy-wide level of investment and let $S$ be the economy-wide level of savings, thus $S = I$. The level of savings in the economy is that portion of income, $Y$, not used for consumption. That fraction we refer to as the savings rate and we will label it, $s$ (lower case). Note that as
a fraction of income, $0 < s < 1$. Thus, $I = sY$. Investment is the fraction of income saved. On a per capita basis we would write: $i = sy$. Using the per capita production function then we have that:

$$i = sAk^{1/3}$$

which states that gross investment per capita, $i$, is the savings rate, $s$, times the income per capita ($y$) which is determined by the technology level, $A$, and the capital-labor ratio, $k$. Replace $i$ in equation (5) with the expression above (7) and we have:

$$\Delta k = sAk^{1/3} - (n + \delta)k.$$  

The above equation (8) has the same interpretation as the equation in (6). To visualize this equation and the relationship between $k$ and $y$ consider Figure 3 below.

The upper curve is the per capita production function given in (3), the lower curve is gross investment which, as we just discussed is the savings rate times per capita income which is the first term on the right-hand side of (8). The straight line beginning at the origin represents effective depreciation $(n + \delta)k$. The capital-labor ratio, $k$, is on the x-axis and we can see the levels of $i$ and $y$ on the y-axis. The difference between income and investment per capita, $y - i$, is consumption per capita, $c$, since the difference is income minus savings or the fraction of income used in expenditures. The vertical difference between the $sAk^{1/3}$ (gross investment) and $(n+\delta)k$ (effective depreciation) is net investment, $\Delta k$. 

![Figure 3: Solow Growth Model](image)
Now, consider the level of the capital stock shown in the figure as $k^*$. At that point the curve for gross investment and the line for effective depreciation intersect such that $sA_k^{1/3} = (n + \delta)k$. What that means is that the level of investment in the economy is just sufficient to offset depreciation. Therefore net investment, $\Delta k$, is zero as can be seen from equation (8) and the capital-labor ratio is unchanged. We refer to that level of the capital-labor ratio as the steady-state level. All the variables with an asterisk ($k^*$, $i^*$, $c^*$, and $y^*$) indicate the levels of those variables when the economy is at the steady-state point. Since income per capita, $y$, is directly tied to the capital-labor ratio, when the capital-labor ratio is unchanging, GDP per capita is unchanging, i.e. standards of living do not change. Restated another way at $k^*$ there is no growth in GDP per capita and thus $y^*$ is not changing.

Consider the economy where the capital-labor ratio is not at $k^*$, but at a different level, $k_0$ in the figure. At that level of the capital-labor ratio, gross investment, $i$, is measured on the savings curve. Effective depreciation is the vertical distance up to the $(n + \delta)k$ line. Thus, net investment is the gap between the depreciation line and the savings curve. GDP per capita, $y$, is found along the upper most curve - the per worker production function. The difference between the per worker production function and the savings curve is $y - i$, which is the level of per capita consumption, $c$.

**THOUGHT EXERCISE 3:** Suppose there is no population growth, $n = 0$. How does equation (8) change? Construct a diagram for this case similar to Figure 3. How does it differ from Figure 3?

When the economy is not at $k^*$, then we have economic growth. To see this look at Figure 4 below.

*Figure 4: Behavior of Economy Away from Steady-State*
In this figure, I have removed the graph of the per-worker production function \( y = A k^{1/3} \) just to make it more clear. First consider a situation where an economy has a low level of the capital-labor ratio represented by \( k_{LOW} \). From the graph we can see that \( sA k^{1/3} \) lies above \((n+\delta)k\), thus \( \Delta k > 0 \). In words, net investment is positive and the capital-labor ratio is increasing (growing). If \( k \) is growing it means that the economy is moving from \( k_{LOW} \) towards the right as \( k \) increases over time. Since \( k \) is growing then from the per-worker production function, \( y \), is also increasing meaning that GDP per capita is rising.

On the other hand, if \( k \) is larger than \( k^* \), represented by \( k_{HIGH} \) in the figure, then the depreciation of the capital exceeds the gross investment level. Therefore \( \Delta k < 0 \). That is, the capital-labor ratio is declining and the economy is moving to the left along the x-axis as the capital-labor ratio shrinks. GDP per capita is falling. Whether we are at points like \( k_{LOW} \) or \( k_{HIGH} \) the economy is moving over time towards the steady-state level \( k^* \).

Example 1. Suppose a country suffers a major earthquake which destroys much of its capital stock (e.g. Tokyo earthquake of 1923). The capital-labor ratio falls significantly and is well below the steady-state level. Over time, the country invests such that it rebuilds its capital stock and the capital-labor ratio rises. Note that initially, the growth rate itself is high but declines as \( k \) increases until the growth rate of the economy reaches zero once it returns to \( k^* \). The growth rate of the capital-labor ratio is given by:

\[
\text{Growth rate of capital-labor ratio} = \frac{\Delta k}{k}
\]

which is just the percentage growth rate formula. Divide equation (8) through by \( k \) to get:

\[
\frac{\Delta k}{k} = \frac{sA}{k^{2/3}} - (n + \delta).
\]

The decline in the growth rate occurs for two reasons. First, as the economy grows, because of diminishing returns to the capital-labor ratio the rate at which output rises is declining. This effect is captured in the first term on the right-hand side \( sA/k^{2/3} \). Since the capital-labor ratio is in the denominator, as the capital-labor ratio increases this number grows smaller. Second, the effective depreciation rate is a constant fraction \((n + \delta)\) of the capital-stock. As \( k \) increases a larger and larger amount of gross investment must go towards replacing worn out equipment. Taken together the level of net investment declines as the capital-labor ratio increases. Thus, the growth rate of the capital-stock \( \Delta k/k \) declines as \( k \) rises. The same logic applies to GDP per capita, \( y \). So, the earthquake causes GDP per capita to drop immediately with the destruction of the capital stock which lowers the capital-labor ratio. Once the economy begins to invest again, the GDP growth rates will be very small.

\footnote{\( \frac{\Delta k}{k} \) represents the same idea we used for the one year growth rates for GDP and for prices. As an example, \( \frac{\Delta k}{k} \) for, say, the year 2005, would be \( \frac{\Delta k}{k} = (k_{2005} - k_{2004})/k_{2004} \).}
high initially and decline over time as it returns to the steady-state level of the capital-labor ratio, $k^\ast$.

**Example 2.** An example of a situation where an economy is at $k_{HIGH}$ would be the black plague of Europe in the 1300s which killed approximately 30% of Europe’s population (and some estimates put the number as high as 60%), but leaving the capital stock intact. With too few people saving and investing, the capital-labor ratio declined towards the steady state level of $k^\ast$ which could be supported by the lower level of the population. As the capital-labor ratio falls, GDP per capita declines as well. The rate at which the decline (negative growth) occurs is initially very large but the decline slowed down as Europe approached the new steady-state level of the capital-labor ratio.

**THOUGHT EXERCISE 4:** Consider two modern day countries in light of the model. Look at Figure 5 below. The first country is a developing nation with a very low level of the capital stock. The second country is an industrialized nation but its capital-labor ratio is slightly lower than the steady-state level of the capital-labor ratio, $k^\ast$. Which country grows faster and why?

![Figure 5: Convergence](image)

The above example in the thought exercise illustrates the concept of **Unconditional Convergence**.

**Definition 1** *Unconditional Convergence:* Poor economies will eventually catch up to the rich countries such that living standards around the world will be the same in the long-run.
That follows from the idea that economies with levels of the capital-labor ratio farther away from \( k^* \) and will grow faster and thus catch up. That is a very strong prediction, but note the term “unconditional.” It assumes that other factors, besides the current level of the capital-labor ratio, are nearly the same such as access to technology, government policies, savings rates, and population growth rates among other things. Therefore, a different prediction from the Neo-classical growth model is termed **Conditional Convergence**.

**Definition 2 **Conditional Convergence: **Economies with similar characteristics (e.g. savings rates, technology) will converge to similar standards of living.**

Conditional convergence says that if we identify a group of countries with similar characteristics, (e.g. UK, Italy, Germany) they will converge to similar standards of living. So if, say, Italy’s capital-labor ratio is significantly lower than in the UK (as it was in the early 1960s) but other factors that affect growth are close to being the same, then we would expect Italy to grow faster than the UK and Italy’s GDP per capita and capital-labor ratio will eventually be nearly the same as the UK’s. On the other hand, suppose we compare the UK to Kenya where savings rates, access to technology, and population growth rates all differ widely. Then, even though Kenya’s capital-labor ratio is much lower than in the UK, we would not necessarily expect Kenya to grow faster and catch up to the UK.

To illustrate how these other factors affect growth and standards of living, consider 2 fictional countries: Papadakistan (PPS for short) and Tsetsekostan (TTS for short). Suppose the countries are completely identical except that the savings rate in PPS, \( s_{PPS} \), is much lower than in TTS, \( s_{TTS} \). Look at Figure 6 below.
Because TSS has a higher savings rate the curve showing the relationship between the capital-labor ratio, \( k \), and gross investment, \( i \), is shifted proportionately upwards for TSS. Thus, the intersection between the effective depreciation line, \((n+\delta)k\) and savings (gross investment), \( sy = sAk^\alpha \), lies much further to the right at a higher level of the capital-labor ratio. Thus, TSS will achieve a higher GDP per capita in long-run no matter where their initial levels of the capital-labor ratio are currently. Suppose, as shown in the diagram, that the current levels of the capital-labor ratios in PSS and TSS are such that \( k_{PSS} < k_{TSS} \). If the countries were completely identical including the savings rate then the prediction of unconditional convergence would hold and we would expect PSS to grow faster because it is farther from \( k^* \), that is, PSS has more room to grow. However, since the savings rate in PSS is much lower, even when \( k_{PSS} < k_{TSS} \) PSS may or may not grow faster than TSS. Their growth rates will depend on how far from their own \( k^* \) levels they currently are.

**THOUGHT EXERCISE 5:** Construct a diagram for PSS and TSS where they are identical in every way (including the savings rate), but the technology level, \( A \), is much higher in PSS than in TSS. Which country will have a higher steady-state level of GDP per capita? If they both begin with the same initial capital-labor ratio, which country grows faster? Explain why.

**THOUGHT EXERCISE 6:** Construct another diagram for PSS and TSS where they are identical in every way, but now suppose the population growth rate is much higher in PSS than in TSS. Which country will have a higher steady-state? If they both begin with the same initial capital-labor ratio, which country grows faster? Explain why.

*Important reminder - at the steady-state level of the capital-labor ratio, \( k^* \), the growth rate of GDP per capita is zero because the economy is saving just enough to offset the effective depreciation.*

However, for the past 150 years we have had positive levels of the growth rate of GDP per capita. Standards of living have been steadily increasing and we have not observed the US economy and most other economies flattening out such that GDP per capita stays the same over a long period of time. Notice that while increases in the savings rate will increase the steady-state level of GDP per capita, higher \( s \) cannot sustain growth because the savings rate cannot be greater than 1. (Can you think of any other problems associated with government policies that try to raise the savings rate to increase GDP per capita?)

In contrast, the technology parameter, \( A \), has no known upper bound. There is of course an argument that there does exist limits to our knowledge/technology and that argument has been made for nearly 200 years. At this point, its not clear whether that limit exists and/or how far away from that limit we actually are.

Consider the US economy which has been at the forefront of R&D investment for most of the past 60-70 years. As production becomes more efficient...
through the application of technology, our productivity rises and is reflected in an increase in \( A \). Figure 7 illustrates the effect.

**Figure 7: Effect of Technological Improvements**

![Graph showing the effect of technological improvements on savings and GDP per capita.](image)

The lower of the two savings curves is for the US as of 1960 and the upper savings curve is for 2000. The reason the savings curves differ across the years is that in 1960 our productivity level was \( A_1 \) and in 2000 it was \( A_2 \), where \( A_2 > A_1 \), that is, technology improved. The steady-state capital-labor ratio and GDP per capita in 1960 were much lower than in 2000. Thus, if the US capital-labor ratio in 1960 was close to the steady-state level, as the technology improved, the savings curve shifted up. As the savings curve shifts up, the steady-state level of the capital-labor ratio increases and there is growth in GDP per capita and standards of living rise. The more productive the economy becomes through raising \( A \), the further out the steady-state level of the capital-labor ratio moves. If technology continually improves, we never reach the steady-state as the economy is always chasing it. Continued technological advances and other improvements that make the economy more productive sustain positive growth rates in GDP per capita and standards of living.

**THOUGHT EXERCISE 7:** Which government policy would be more likely to raise GDP per capita growth rates and why? A) A consumption (sales) tax to discourage consumption and increase savings; or B) subsidies for R&D paid for by taxing corporate profits. Explain the effects of each on growth rates. Distinguish between temporary effects and permanent effects.

**THOUGHT EXERCISE 8:** The case of Singapore and Hong Kong. Singapore and Hong Kong are considered two of the “East Asian Growth Miracles.” Both had very high growth rates in the post-World War II era. Their growth rates were largely the same since 1960. If both countries started at
roughly the same distance from the steady-state, what would you expect of
t heir growth rates over time? Use a diagram to illustrate your answer and
relate your answer to the concepts of “unconditional convergence” and “condi-
tional convergence”. When looking more carefully at the data, we see that the
savings rate in Singapore was considerably higher than in Hong Kong. If that
were the only difference between the two how would you expect their growth
rates to compare? Use diagrams to illustrate your answer. Since their growth
rates, at least until recently were largely the same (and assuming the population
growth rates, \( n \), and depreciation \( \delta \) were the same), how could you explain them
having very similar growth rates (and similar initial and recent levels of GDP
per capita)? Based on your explanation, what would you expect for the future
of these two countries?