ENGINEERING RELIABILITY

FAILURE MODELS

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INTRODUCTION

FAILURE RATE
- Time to Failure
- Reliability Function
- Failure Rate
- MTTF & MRL

CONSTANT RATE MODELS
- The Exponential Distribution
- Repeated Demand

VARIABLE RATE MODELS

SUMMARY
We will be concerned with the following measures of a product or system that is not repaired upon failure.

- The reliability (or survivor) function, $R(t)$.
- The failure rate, $\dot{z}(t)$ or $\lambda(t)$.
- The mean time to failure, $MTTF$.
- The mean residual life $MRL$.
- Constant rate models.
The failure time $T$ of a product or systems is a random variable. Time can take on different meanings depending on the context, e.g.,

- Calendar time.
- Operational time.
- Distance driven by a vehicle.
- Number of cycles for a periodically operated system.
- Number of times a switch is operated.
Probability Characterization of Failure Time

Associated with the time to failure $T$ is the probability function

$$F(t) = P(T \leq t)$$

which is the probability that the system fails within the time interval $(0, t]$. If $T$ is a continuous random variable, the probability function is related to its probability density function $f(t)$ by

$$F(t) = \int_0^t f(\tau) d\tau$$
The lognormal distribution has been found useful in the failure analysis of items subjected to repeated loadings.

while the normal distribution is ideal for characterizing the influence of the sum of a large number of independent events, the lognormal is appropriate for characterizing the product of a large number of independent events.

\[ f(t) = \frac{1}{\sqrt{2\pi \sigma t}} e^{-\frac{1}{2} \left( \frac{\ln t - \mu}{\sigma} \right)^2}, \quad t > 0 \]

\[ F(t) = \frac{1}{2} \left( 1 + \text{Erf} \left( \frac{\ln t - \mu}{\sqrt{2} \sigma} \right) \right), \quad t > 0 \]
The reliability function is:

\[ R(t) = P(T > t) = 1 - F(t) = \int_t^{\infty} f(\tau) \, d\tau \]

- \( R(t) \) is the probability that the item will not fail in the interval \((0, t]\).
- \( R(t) \) is the probability that it will survive at least until time \( t \) – it is sometimes called the survival function.
Consider the conditional probability:

\[ P(t < T \leq t + \Delta t | T > t) = \frac{P(t < T \leq t + \Delta t)}{R(t)} = \frac{F(t + \Delta t) - F(t)}{R(t)} \]

The failure rate (or, hazard function) is defined as:

\[ \lambda(t) = \lim_{\Delta t \to 0} \frac{P(t < T \leq t + \Delta t | T > t)}{\Delta t} = \frac{f(t)}{R(t)} \]

\( \lambda(t) \, dt \) is the probability that the system will fail during the period \((t, t + dt]\), given that it has survived until time \(t\).
Suppose the failure rate $\lambda(t)$ is known. Then it is possible to obtain $f(t)$, $F(t)$, and $R(t)$

$$f(t) = \frac{dF(t)}{dt} = -\frac{dR(t)}{dt} \Rightarrow \lambda(t) = -\frac{dR/dt}{R}$$

$$\frac{dR}{R} = -\lambda(t)\,dt$$

$$R(t) = \exp\left[-\int_0^t \lambda(\tau)\,d\tau\right]$$

$$f(t) = \lambda(t)\exp\left[-\int_0^t \lambda(\tau)\,d\tau\right]$$

$$F(t) = 1 - \exp\left[-\int_0^t \lambda(\tau)\,d\tau\right]$$
Example – TV Sets

Example (TV Set Failure Data)

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<tr>
<th>Day</th>
<th>Failures</th>
<th>Failure Rate</th>
</tr>
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<tbody>
<tr>
<td>1</td>
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<td>0.018</td>
</tr>
<tr>
<td>2</td>
<td>12</td>
<td>0.012</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>0.010</td>
</tr>
<tr>
<td>4</td>
<td>7</td>
<td>0.007</td>
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<td>6</td>
<td>0.006</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>0.005</td>
</tr>
<tr>
<td>7</td>
<td>4</td>
<td>0.004</td>
</tr>
<tr>
<td>8</td>
<td>3</td>
<td>0.003</td>
</tr>
<tr>
<td>9</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>0.001</td>
</tr>
</tbody>
</table>
Example – TV Sets, Cont’d

\[ \lambda(t) = 0.02t^{-0.56} \quad f(t) = 0.02t^{-0.56} e^{-0.04545t^{0.44}} \]
Weibull Distribution

Perhaps the most frequently used distribution to model time to failure probabilities is the Weibull distribution, Weibull(\(\alpha, \beta\)), The probability function is

\[
F(t) = \begin{cases} 
1 - e^{-(\beta t)^\alpha} & t \geq 0 \\
0 & t < 0 
\end{cases}
\]

and the corresponding density function is

\[
f(t) = \frac{d}{dt} F(t) = \begin{cases} 
\alpha \beta \alpha t^{\alpha-1} e^{-(\beta t)^\alpha} & t \geq 0 \\
0 & t < 0 
\end{cases}
\]
The Reliability function is:

\[ R(t) = 1 - F(t) = e^{-(\beta t)^{\alpha}}, \quad t > 0 \]

and the failure rate function is

\[ \lambda(t) = \frac{f(t)}{R(t)} = \alpha \beta t^{\alpha-1}, \quad t > 0 \]
Mean Time to Failure

The mean time to failure is

\[ MTTF = E\{T\} = \int_0^\infty t f(t) \, dt \]

note that

\[ f(t) = \frac{d}{dt} F(t) = -\frac{d}{dt} R(t) \]

from which

\[ MTTF = -\int_0^\infty t \frac{dR(t)}{dt} \, dt = -tR(t)|_0^\infty + \int_0^\infty R(t) \, dt \]

\[ MTTF = \int_0^\infty R(t) \, dt \]
Consider a system with reliability function

\[ R(t) = \frac{1}{(0.2t + 1)^2}, \text{ for } t > 0 \quad (t \text{ in months}) \]

- probability density \( f(t) = -\frac{d}{dt}R(t) = \frac{0.4}{(0.2t+1)^3} \)
- failure rate \( \lambda(t) = \frac{f(t)}{R(t)} = \frac{0.4}{(0.2t+1)} \)
- mean time to failure \( MTTF = \int_0^\infty R(t) \, dt = 5 \text{ months} \)
Mean Residual Life

An item begins operation at time 0 and is still operating at time $t$. We wish to compute the probability that it will survive an additional interval of length $\tau$. This is the conditional reliability function at age $t$.

$$R(\tau | t) = P(T > t + \tau | T > t) = \frac{P(T > t + \tau)}{P(T > t)} = \frac{R(t + \tau)}{R(t)}$$

The mean residual (or, remaining) life at age $t$ is

$$MRL(t) = \int_0^\infty R(\tau | t) \, d\tau = \frac{1}{R(t)} \int_t^\infty R(\tau) \, d\tau$$
Consider an item with failure rate $\lambda(t) = t/(t + 1)$ Now compute

$$R(t) = \exp \left( - \int_0^t \frac{\tau}{\tau + 1} d\tau \right) = (t + 1) e^{-t}$$

$$MTTF = \int_0^\infty (t + 1) e^{-t} dt = 2$$

The conditional reliability function is

$$R(\tau|t) = P(T > \tau + t|T > \tau) = \frac{(t + \tau + 1) e^{-(t+\tau)}}{(t + 1) e^{-t}} = \frac{t + \tau + 1}{t + 1}$$

So,

$$MRL = \int_0^\infty R(\tau|t) d\tau = 1 + \frac{1}{t + 1}$$
**Example: Weibull Distribution**

The mean time to failure is

$$MTTF = \int_0^\infty R(t) \, dt = \frac{1}{\beta} \Gamma\left(\frac{1}{\alpha} + 1\right)$$

The remaining life is

$$MRL = \frac{1}{R(t)} \int_t^\infty R(t) \, dt = \left(\frac{\alpha}{\beta}\right) e^{\left(\frac{t}{\beta}\right)\alpha} \Gamma\left(\frac{1}{\alpha}, \left(\frac{t}{\beta}\right)^\alpha\right)$$

The median life is

$$R(t_m) = 0.50 \implies t_m = \frac{1}{\beta} (\ln 2)^{1/\alpha}$$

The variance of $T$ is

$$\text{var}(T) = \frac{1}{\beta^2} \left( \Gamma\left(\frac{2}{\alpha} + 1\right) + \Gamma^2\left(\frac{1}{\alpha} + 1\right) \right)$$
Suppose $\lambda(t) = \lambda_0$, a constant. Then,

$$R(t) = e^{-\lambda_0 t}, \quad F(t) = 1 - e^{-\lambda_0 t}, \quad f(t) = \lambda_0 e^{-\lambda_0 t}$$

This is the exponential distribution. We can easily compute

$$\mu = MTTF = \frac{1}{\lambda_0}$$

$$\sigma = \frac{1}{\lambda_0}$$

$$R(\mu) = 0.368, \quad F(\mu) = 0.632$$
Example

Suppose $\lambda = .02 \ hr^{-1}$,

- What is the probability of a failure in the first 10 hours of service?

  \[
P(T \leq 10) = F(10) = 1 - e^{0.02 \times 10} = 0.181
  \]

- Suppose the unit operates satisfactorily for the first 100 hours. What is the probability of failure in the next 10 hours?

  - Let $X =$ event that the unit operates for 100 hours
    \[P(X) = R(100) = .135\]
  - Let $Y =$ event that the unit fails within 110 hours
    \[P(Y) = F(110) = .1108\]
  - We want to compute $P(Y|X)$. By definition
    \[
P(Y|X) = \frac{P(Y \cap X)}{P(X)} = \frac{P(100 \leq t \leq 110)}{P(X)} = \frac{F(110) - F(100)}{R(100)} = 0.181
    \]
Suppose a unit is subjected to repeated demand (e.g., engine startup) over a large time interval \((0, t]\):

- \(p\) denotes the probability of failure to respond to a demand,
- the response to each demand is an independent event,
- \(n\) denotes the number of demands in time \(t\),
- \(m = n/t\) denotes the average number of demands per unit time,

The probability of \(k\) failures in \(n\) demands is given by the Binomial distribution

\[
b(k; n, p) = C^n_k p^k (1 - p)^{n-k}.
\]

Let \(N\) denote the number of trials until the first failure. Thus, \(N = n\) means that the first \(n - 1\) trials are successful, and failure occurs at trial \(n\). The distribution of \(N\) is the geometric distribution

\[
P(N = n) = (1 - p)^{n-1} p
\]
The reliability function, $R(n)$, is the probability that the first failure occurs for some trial $N > n$. Consequently,

$$R(n) = P(N > n) = b(0; n, p) = (1 - p)^n$$

Suppose the single trial failure probability $p$ is small and the length of the observed sequence $N$ large, in fact $Np = \lambda$, $\lambda = \text{constant}$. Then,

$$\lim_{p \to 0} (1 - p)^{\frac{\lambda}{p}} = e^{-\lambda} \Rightarrow R(n) = e^{-np} = e^{-mpt} = e^{-\lambda_0 t}$$

where $\lambda_0 = mp$ is the equivalent failure rate.
MULTIPLE PERFORMANCE LEVELS

A unit operates at different performance levels during cyclic operation. In each operating phase the unit fails at constant rate.

EXAMPLE

- a motor cycles through 3 phases: start, run, standby,
- \( N \), number of starts per service cycle,
- \( T \) denotes the number of service cycles to failure,
- \( c \), time fraction motor runs during a cycle,
- \( 1 - c \), time fraction motor is in standby,
- \( p \), probability of failure to start,
- \( \lambda_r \), failure rate in run state,
- \( \lambda_s \), failure rate in standby state.

\[
\lambda_c \triangleq \lambda_d + c\lambda_r + (1 - c)\lambda_s, \quad \text{where} \quad \lambda_d \triangleq Np
\]

\[
R(t) = e^{-\lambda_c t}
\]
COMMON VARIABLE RATE MODELS

- Time to failure is typically described by normal, lognormal or Weibull probability distributions.
- The corresponding failure rates can be computed from

\[ \lambda(t) = -\frac{1}{R(t)} \frac{dR(t)}{dt} \]

**Normal**: \[ \lambda(t) = \frac{\phi(z)}{\sigma (1 - \Phi(z))} \]; \[ z = \frac{t - \mu}{\sigma} \]

**Lognormal**: \[ \lambda(t) = \frac{\phi(z)}{\sigma t \Phi(z)} \]; \[ z = \frac{\ln(t) - \mu}{\sigma} \]

**Weibull**: \[ \lambda(t) = (\alpha \beta) (\beta t)^{\alpha - 1} \]
The Wear-in Failure Mode & the Proof Test

- When initial failure rates are high, testing can improve the reliability of deployed product.
- An initial, short period, $t_p$, of testing of the entire batch weeds out faulty product.
- Suppose $\lambda(t) = at^{-b}$, $a, b > 0$, then we can compute

$$R(t - t_p | t_p)$$

$$R(\tau | t_p) = \frac{R(t_p + \tau)}{R(t_p)} = \frac{e^{-\int_{t_p}^{t_p+\tau} \lambda(\xi) d\xi}}{e^{-\int_{0}^{t_p} \lambda(\xi) d\xi}} = e^{-\int_{t_p}^{t_p+\tau} \lambda(\xi) d\xi}$$
Time to failure

Failure rate

Computation of probability functions from failure rate

definitions of mean time to failure (MTTF) and remaining life (MRL)

Introduced lognormal, exponential and Weibull distributions

Examples of constant failure rate problems

Proof Testing