ENGINEERING RELIABILITY

FUNDAMENTALS OF PROBABILITY

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OUTLINE

PRELIMINARY DEFINITIONS

THE PROBABILITY FORMALISM

Events
Probabilities

COMBINATORICS

Elementary Combinatorics
Bernoulli Trials
Binomial & Poisson Distributions

RANDOM VARIABLES

Discrete & Continuous RV’s
Normal Distribution
Bayes’ Rule Revisited

SUMMARY
A random variable (denoted by $X$) is a variable that can assume one or more possible numerical values (denoted by $x$).

The value $x$ that $X$ assumes is determined by chance.

A random variable may be:

- **discrete**
  
  $$x \in \{1, 2, 3\}$$

- **or continuous**
  
  $$x \in \{x \mid 0 \leq x < \infty\}$$
for discrete random variables the probability distribution function is the set of probabilities

\[ f(x) = P(X = x) , \quad \text{and} \quad \sum_x f(x) = 1 \]

for continuous random variables the (cumulative) probability function is the function:

\[ F(x) = P(X \leq x) \]

for continuous random variables the probability distribution or probability density is:

\[ f(x) = \frac{d}{dx} F(x) , \quad \text{and} \quad \int_{-\infty}^{\infty} f(x) \, dx = 1 \]
Let the random variable $T$ denote the time of failure of a product during service. Then the possible values of $T$ are the set $0 < t < \infty$.

The associated probability density is $f(t)$, the probability that failure occurs at time $t$ is

$$F(t) = \int_0^t f(\tau) d\tau$$

The reliability function is the probability of survival to time $t$

$$R(t) = P(T \geq t) = 1 - F(t) = \int_t^\infty f(\tau) d\tau$$
**Sample Space**

The underlying idea is that there is a well-defined trial and a set of possible outcomes.

**Example (Flipping a Coin 3 Times)**

Flipping a coin 3 times yields the following set of outcomes:

\[
\begin{align*}
\xi_1 &= TTT \\
\xi_2 &= TTH \\
\xi_3 &= THT \\
\xi_4 &= THH \\
\xi_5 &= HTT \\
\xi_6 &= HTH \\
\xi_7 &= HHT \\
\xi_8 &= HHH
\end{align*}
\]

- **Sample space**, \( S \), the set of all possible outcomes.
- **Elementary outcome**, \( \xi \), the individual elements of \( S \).
**Events**

**Event** $A$: a subset of the sample space. In general, an event is defined by a proposition about the elements in it.

**Example (Flipping a coin 3 times)**

$A$ is the event that a tail shows up on the second toss

$$A = \{\xi_1, \xi_2, \xi_5, \xi_6\}$$

**Elementary event**: $\{\xi\}$, where $\xi$ is an elementary outcome.

**Sure event**: the entire sample space, $S$.

**Impossible event**: the empty set, $\emptyset$.

**Complementary event**: $A^c$ consists of all events not in $A$.

**Mutually Exclusive events**: events that have pairwise empty intersections.
A probability measure is a function that assigns a ‘likelihood’ of occurrence to each subset of $S$ (to each event)

A Probability Measure $P$ is a function on the set of subsets of $S$ that has the following properties:

$p(S) = 1$

$p(A) \geq 0$ for each $A \subset S$

For any sequence of mutually exclusive events $A_1, A_2, \ldots$

$$p\left( \bigcup_{i=1}^{\infty} A_i \right) = \sum_{i=1}^{\infty} p(A_i)$$
The Venn diagram is an aid in visualizing basic properties of sets. It can also serve as a simple visual representation of the probability model.

- The sample space $S$ is visualized as a rectangular set of points in the plane.
- The elementary outcomes are the points in $S$.
- The probability is associated with a the (non-uniform) distribution of a unit mass over the set $S$. In the case of a finite number of outcomes, the mass is concentrated at a finite number of points.
VENN DIAGRAMS
BASIC SET OPERATIONS

PRELIMINARY DEFINITIONS

THE PROBABILITY FORMALISM

EVENTS

PROBABILITIES

COMBINATORICS

ELEMNTARY COMBINATORICS
BERNOULLI TRIALS
BINOMIAL & POISSON DISTRIBUTIONS

RANDOM VARIABLES

DISCRETE & CONTINUOUS RV'S
NORMAL DISTRIBUTION
BAYES' RULE REVISITED

SUMMARY
Conditional Probability & Independent Events

The probability that an event $A$ occurs, given the occurrence of an event $B$ is called the conditional probability of $A$ given $B$. It is denoted $P(A | B)$. From the Venn diagram we see that,

$$P(A \cap B) = P(A | B) \cdot P(B)$$

$$P(A | B) = \frac{P(A \cap B)}{P(B)}, \quad \text{if } P(B) \neq 0$$

Two events $A$ and $B$ are independent if the occurrence of one does not ‘condition’ the occurrence of the other, i.e.,

$$P(A | B) = P(A) \quad \text{and} \quad P(B | A) = P(B)$$

Thus, for independent events

$$P(A \cap B) = P(A) \cdot P(B)$$
**Independent Events**

**Example (Flipping a Coin 3 Times)**

- reconsider the coin flipping experiment with sample space $S$ shown below.
- let $A$ be the event that a $T$ occurs on the third toss.
- let $B$ be the event that an $H$ occurs on the second toss.

\[
P(A) = \frac{1}{2}, \quad P(B) = \frac{1}{2}, \quad P(A|B) = \frac{1}{2}, \quad P(B|A) = \frac{1}{2}
\]

Notice that the events are independent, but not mutually exclusive.

<table>
<thead>
<tr>
<th>$\xi_1$</th>
<th>$\xi_2$</th>
<th>$\xi_3$</th>
<th>$\xi_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$TTT$</td>
<td>$TTH$</td>
<td>$THT$</td>
<td>$THH$</td>
</tr>
<tr>
<td>$HTT$</td>
<td>$HTH$</td>
<td>$HHT$</td>
<td>$HHH$</td>
</tr>
</tbody>
</table>

Diagram:

- $\xi_1 = TTT$
- $\xi_2 = TTH$
- $\xi_3 = THT$
- $\xi_4 = THH$
- $\xi_5 = HTT$
- $\xi_6 = HTH$
- $\xi_7 = HHT$
- $\xi_8 = HHH$

Summary:

- $A$
- $B$
- $S$
**Bayes’ Rule**

- **Expansion rule:** $A \subset \bigcup_{i \in J} B_i$, events $B_i$ mutually exclusive

  $$P(A) = \sum_{i \in J} P(A|B_i) P(B_i)$$

- **Bayes’ rule:** $A \subset \bigcup_{i \in J} B_i$, events $B_i$ mutually exclusive

  $$P(B_i|A) = \frac{P(B_i \cap A)}{P(A)} = \frac{P(A|B_i) P(B_i)}{\sum_{i \in J} P(A|B_i) P(B_i)}$$
A population of size $n$ is a set of $n$ distinguishable elements.

Consider a population of size $n$ from which we obtain an ‘ordered’ sample of size $r$.

- Sampling with replacement there are $n^r$ possible samples.
- Sampling without replacement there are $(n)_r = n(n-1)\cdots(n-r+1)$ possible samples.
- Set $n = r$ to find that there are $n!$ different orderings of $n$ elements.
Binomial Coefficients

We want to choose a subpopulation of size $r$ from a population of size $n$. How many different such subpopulations are there?

- There are $(n)_r$ samples of size $r$ without replacement,
- Each $r$-sample can be ordered in $r!$ ways,
- Thus, there are $(n)_r / r!$ subpopulations of size $r$.

$$\binom{n}{r} = \frac{(n)_r}{r!} = \frac{n(n-1) \cdots (n-r+1)}{r(r-1) \cdots 1} = \frac{n!}{r!(n-r)!} = C^n_r$$
By a **Bernoulli trial** we mean an experiment consisting of a sequence of independent trials with two possible outcomes, *success* and *failure*, having probability of failure $p$ and probability of success $q = 1 - p$.

**Fundamental Problem**: Consider a Bernoulli trial of length $n$. What is the probability of exactly $k$ failures?

The event ‘$n$ trials results in $k$ failures and $n - k$ successes’ can happen in as many ways as $k$ letters $F$ can be distributed among $n$ places.

In other words, i.e., how many subpopulations of size $k$ can be constructed from a population of size $n$? The event consists of:

$$C_k^n$$ points, each with probability $p^k q^{n-k}$
The probability that $n$ Bernoulli trials with probability $p$ for failure and $q = 1 - p$ for success results in $k$ failures and $n - k$ successes is given by the Binomial distribution:

$$b(k; n, p) = \binom{n}{k} p^k q^{n-k}, \quad 0 \leq k \leq n$$
Poisson Approximation

If

- the probability of failure $p$ is small,
- the number of trials $n$ is large, so that $np = \lambda$, a constant

then a good approximation to the Binomial distribution is the **Poisson distribution**:

$$ p(k; \lambda) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad \lambda = np $$
**Random Variables**

A function $X(\cdot)$ that maps a sample space $S$ to the real numbers $R$ is called a (real valued) random variable if it has the property:

$$\{\xi \in S | X(\xi) \leq x\} \text{ is an event } \forall x \in R$$

All elementary outcomes that result in $X(\xi) \leq x$ is a valid subset of $S$, for all real $x$.

▶ A random variable is discrete if it can assume a finite set of distinct values, say $x_i, i = 1, \ldots, n < \infty$

▶ A random variable is continuous if the values it can assume are continuously distributed over its range, say $-\infty < x < \infty$
Discrete random Variables

- $f(x_i) = P(X = x_i)$ is called the probability distribution,
  note: $\sum_i f(x_i) = 1$

- The (cumulative) probability function is:
  $F(x_k) = \sum_{i=1}^k f(x_i)$
  note: $F(x_k) = P(X \leq x_k)$

- Mean: $\mu = \sum_{i=1}^n x_i f(x_i)$

- Variance: $\sigma^2 = \sum_{i=1}^n (x_i - \mu)^2 f(x_i)$

- Standard deviation: $\sigma$
Coin Flipping Example

Example (Flipping a coin 3 times)

- Flipping a coin 3 times yields a set of 8 outcomes.
- Assume: on a single toss, \( P(H) = P(T) = \frac{1}{2} \).
- Define: \( X = \) number of tails in 3 tosses.

\[ \begin{align*}
\xi_1 &= TTT \\
\xi_2 &= TTH \\
\xi_3 &= THT \\
\xi_4 &= THH \\
\xi_5 &= HTT \\
\xi_6 &= HTH \\
\xi_7 &= HHT \\
\xi_8 &= HHH
\end{align*} \]

\[ S \]

\[ X \]

\[ R \]

\[ R \]
CONTINUOUS RANDOM VARIABLES

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Revisited

SUMMARY

➤ the (cumulative) probability function is:
\[ F(x) = P(X \leq x) \]

➤ the probability distribution (density) is:
\[ f(x) = \frac{d}{dx} F(x) \Rightarrow \int_{-\infty}^{\infty} f(x) \, dx = 1 \]

➤ mean: \[ \mu = \int_{-\infty}^{\infty} x f(x) \, dx \]

➤ variance: \[ \sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) \, dx \]

➤ standard deviation: \[ \sigma \]
**Continuous Random Variables, Cont’d**

**Engineering Reliability**

**Preliminary Definitions**

**The Probability Formalism**

- **Events**
- **Probabilities**

**Combinatorics**

- **Elementary Combinatorics**
- **Bernoulli Trials**
- **Binomial & Poisson Distributions**

**Random Variables**

- **Discrete & Continuous RV’s**
- **Normal Distribution**
- **Bayes’ Rule Revisited**

**Summary**

- **median**: $x_m$, $F(x_m) = \int_{-\infty}^{x_m} f(x) \, dx = \frac{1}{2}$
- **mode**: $x_{mode}$, $f(x_{mode}) \geq f(x)$
- **skewness**: $sk = \frac{1}{\sigma^3} \int_{-\infty}^{\infty} (x - \mu)^3 f(x) \, dx$

**Comments on skewness**:

- $sk > 0 \Rightarrow$ left – skewed: $x_{mode} < x_m < \mu$
- $sk < 0 \Rightarrow$ right – skewed: $x_{mode} > x_m > \mu$
- $sk = 0 \Rightarrow$ symmetric: $x_{mode} = x_m = \mu$
**Example – Poisson Distribution**

\[ f(k) = e^{-\lambda} \frac{\lambda^k}{k!} \]

\[ F(k) = \sum_{i=1}^{k} e^{-\lambda} \frac{\lambda^i}{i!} = \frac{(1 + k) \Gamma (1 + k, \lambda)}{\Gamma (2 + k)} \]

Here is the Poisson distribution for \( \lambda = 2, 4, 8 \) (left to right)
EXAMPLE – POISSON DISTRIBUTION, CONT’D

\[ \mu = \sum_{k=0}^{\infty} k \left( e^{-\lambda} \frac{\lambda^k}{k!} \right) = \lambda \]
\[ \sigma^2 = \sum_{k=0}^{\infty} (k - \mu)^2 \left( e^{-\lambda} \frac{\lambda^k}{k!} \right) = \lambda \]
\[ sk = \frac{1}{\sigma^3} \sum_{k=0}^{\infty} (k - \mu)^3 \left( e^{-\lambda} \frac{\lambda^k}{k!} \right) = \frac{1}{\sqrt[3]{\lambda}} \]
The normal distribution has two key applications in reliability:

- It is a good model for the variability of parameters in batch-manufactured parts.
- It is a good approximation to the ‘wear-out’ time to failure distribution.

\[ f(T) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{T - \mu}{\sigma} \right)^2} \]

where \( \mu \) is the mean time to failure and \( \sigma \) is the standard deviation of the time to failure.
The normal distribution is commonly used that the standard normal distribution has been introduced to facilitate computations. Suppose $X$ is a normally distributed random variable with mean $\mu$ and standard deviation $\sigma$. Consider a new random variable $Z$, related to $X$, by the relation

$$Z = \frac{(X - \mu)}{\sigma}$$

Clearly,

$$P(X \leq x) = P \left( Z \leq \frac{(x - \mu)}{\sigma} \right)$$

Equivalently,

$$F(x) = \Phi \left( \frac{(x - \mu)}{\sigma} \right)$$

where $F(x)$ is the probability function for $X$ and $\Phi(z)$ is that of $Z$. Thus,

$$f(x) = \frac{dF(x)}{dx} = \frac{d\Phi(z)}{dz} \frac{dz}{dx} = \phi(z) \frac{dz}{dx}$$
It follows that

$$\phi (z) = \left[ f (x) \left( \frac{dz}{dx} \right)^{-1} \right]_{x \rightarrow \sigma z + \mu} = \left[ \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{1}{2} \left( \frac{x-\mu}{\sigma} \right)^2} \sigma \right]_{x \rightarrow \sigma z + \mu}$$

So,

$$\phi (z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} z^2}$$

and

$$\Phi (z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2} \zeta^2} d\zeta$$

The standard normal is the normal distribution with zero mean and unit variance. It's usefulness follows from the fact that

$$F (x) = \Phi \left( \frac{(x - \mu)}{\sigma} \right), \quad f (x) = \phi \left( \frac{(x - \mu)}{\sigma} \right)$$
**Example (Flat Panel Monitor)**

A flat pane computer monitor is designed to operate for 10,000 hours. Data shows that 2% fail within 1000 hours and 3.8% fail before 2000 hours. Assuming a normal time to failure distribution, determine the mean time to failure.

\[
F(t) = \int_{-\infty}^{t} f(\tau) d\tau = \frac{1}{2} \left( 1 + \text{Erf} \left( \frac{t - \mu}{\sqrt{2} \sigma} \right) \right)
\]

We can obtain the unknown parameters \( \mu, \sigma \) from the two relations

\[
F(1000) = 0.02, F(2000) = 0.038 \Rightarrow \mu = 8351, \sigma = 3580
\]
Bayes’ Rule – Continuous Random Variables

Suppose $X$ and $Y$ are random variables. The following quantities are defined:

- joint probability distribution is
  \[ F(x, y) = P(X \leq x) \cap P(Y \leq y) \]

- joint probability density function is
  \[ f(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y) \]

- conditional density function for $X$ given $Y$
  \[ f(x | y) = \frac{f(x, y)}{f(y)} \]

Bayes rule for densities
\[ f(x | y) = \frac{f(y | x) f(x)}{\int_{-\infty}^{\infty} f(y | x) f(x) \, dx} \]
Summary

Random variable, probability, reliability
Sample space, events
Conditional probability, Bayes’ rule
Combinatorics, Bernoulli trials
Discrete random variables - Binomial and Poisson distributions
Continuous random variables - Normal distribution
Mean, variance skewness, median, mode