MEM 355 Performance Enhancement of Dynamical Systems

*Frequency Domain Design*

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Outline

• Closed Loop Transfer Functions
  ▪ The gang of four
  ▪ Sensitivity functions

• Traditional Performance Measures
  ▪ Time domain
  ▪ Frequency domain

• Robustness
  ▪ Nyquist
  ▪ Traditional gain/phase margins
Closed Loop Transfer Functions

\[
Y = G_p (U + D_1) \\
U = G_c (R - Y - D_2) \\
E = R - Y
\]
Transfer Functions Con’t

Output

\[ Y(s) = \left[ I + G_p(s)G_c(s) \right]^{-1} G_p(s)G_c(s)R(s) - \left[ I + G_p(s)G_c(s) \right]^{-1} G_p(s)D_1(s) \]

\[ + \left[ I + G_p(s)G_c(s) \right]^{-1} G_p(s)G_c(s)D_2(s) \]

Error

\[ E(s) = \left[ I + G_p(s)G_c(s) \right]^{-1} R(s) - \left[ I + G_p(s)G_c(s) \right]^{-1} G_p(s)D_1(s) \]

\[ + \left[ I + G_p(s)G_c(s) \right]^{-1} G_p(s)G_c(s)D_2(s) \]

Control

\[ U(s) = G_c(s) \left[ I + G_p(s)G_c(s) \right]^{-1} R(s) - G_c(s) \left[ I + G_p(s)G_c(s) \right]^{-1} G_p(s)D_1(s) \]

\[ - G_c(s) \left[ I + G_p(s)G_c(s) \right]^{-1} D_2(s) \]
## Gang of Four

<table>
<thead>
<tr>
<th></th>
<th>Output</th>
<th>Error</th>
<th>Control</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Reference</strong></td>
<td>[\frac{G_p G_c}{1 + G_p G_c}]</td>
<td>[\frac{1}{1 + G_p G_c}]</td>
<td>[\frac{G_c}{1 + G_p G_c}]</td>
</tr>
<tr>
<td><strong>Disturbance</strong></td>
<td>[\frac{G_p}{1 + G_p G_c}]</td>
<td>[\frac{-G_p}{1 + G_p G_c}]</td>
<td>[\frac{-G_p G_c}{1 + G_p G_c}]</td>
</tr>
<tr>
<td><strong>Noise</strong></td>
<td>[\frac{-G_p G_c}{1 + G_p G_c}]</td>
<td>[\frac{G_p G_c}{1 + G_p G_c}]</td>
<td>[\frac{G_c}{1 + G_p G_c}]</td>
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Sensitivity Functions

\[ E(s) = [I + L]^{-1} R(s) - [I + L]^{-1} GD_1(s) + [I + L]^{-1} LD_2(s) \]

where \( L := GK \)

sensitivity function: \( S := [I + L]^{-1} \)

complementary sensitivity function: \( T := [I + L]^{-1} L \)

For SISO systems Bode derived:

\[
\frac{dT/T}{dL/L} = \frac{dT}{dL} \frac{L}{T} = \left\{ -[1 + L]^{-2} L + [1 + L]^{-1} \right\} \frac{L}{[1 + L]^{-1} L} = -[1 + L]^{-1} L + 1 = [1 + L] = S
\]
Traditional Performance ~ Time Domain

- *rise time*, $T_r$, usually defined as the time to get from 10% to 90% of its ultimate (i.e., final) value.
- *settling time*, $T_s$, the time at which the trajectory first enters an $\varepsilon$-tolerance of its ultimate value and remains there ($\varepsilon$ is often taken as 2% of the ultimate value).
- *peak time*, $T_p$, the time at which the trajectory attains its peak value.
- *peak overshoot*, $OS$, the peak or supreme value of the trajectory ordinarily expressed as a percentage of the ultimate value of the trajectory. An overshoot of more than 30% is often considered undesirable. A system without overshoot is ‘overdamped’ and may be too slow (as measured by rise time and settling time).
Traditional Performance ~ Time Domain
Cont’d

Time response parameters

Ideal pole locations

degree of stability, decay rate $1/\alpha$

$\theta = \sin^{-1} \rho$

Ideal region for closed loop poles
Traditional Performance ~ Frequency Domain

Complementary sensitivity

V-22 Osprey altitude control
K=280
Sensitivity Functions, Cont’d

For unity feed back systems:

$S$ is the error response to command transfer function

$T$ is the output response to command transfer function

Suppose $G(s)$ is strictly proper $m < n$, then

$$\lim_{\omega \to \infty} S(j\omega) = \lim_{\omega \to \infty} \frac{1}{1 + G(j\omega)} = 1$$

$$\lim_{\omega \to 0} S(j\omega) = \lim_{\omega \to 0} \frac{1}{1 + G(j\omega)} = \begin{cases} 
0 & \text{type } G \geq 1 \\
\approx c & \text{type } G = 0
\end{cases}$$

$$\lim_{\omega \to \infty} T(j\omega) = \lim_{\omega \to \infty} \frac{G(j\omega)}{1 + G(j\omega)} = 0$$

$$\lim_{\omega \to 0} T(j\omega) = \lim_{\omega \to 0} \frac{G(j\omega)}{1 + G(j\omega)} = \begin{cases} 
1 & \text{type } G \geq 1 \\
\approx c & \text{type } G = 0
\end{cases}$$
Traditional Performance ~ Frequency Domain

Bandwidth Definitions

Sensitivity Function (first crosses $1/\sqrt{2}=0.707\sim-3\text{db}$ from below):

$$\omega_{BS} = \max_v \left\{ v : |S(j\omega)| < 1/\sqrt{2} \quad \forall \omega \in [0, v) \right\}$$

Complementary Sensitivity Function (highest frequency where $T$ crosses $1/\sqrt{2}$ from above)

$$\omega_{BT} = \min_v \left\{ v : |T(j\omega)| < 1/\sqrt{2} \quad \forall \omega \in (v, \infty) \right\}$$

Crossover frequency

$$\omega_c = \max_v \left\{ v : |L(j\omega)| \geq 1 \quad \forall \omega \in [0, v) \right\}$$
Example: Osprey Bandwidth

Complementary sensitivity

Open loop

Sensitivity

MAGNITUDE

dB

rad/sec

0.05 0.1 0.5 1 5 10

0 -5

-10

-15

0 -5

-10

-15

0 0.1 0.5 1 5 10

rad/sec

Mag.

0 -5

-10

-15

0 -5

-10

-15

0 0.1 0.5 1 5 10

rad/sec

Mag.

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Transfer Functions as Maps
Traditional Performance ~ Frequency Domain

Sensitivity peaks are related to gain and phase margin. Sensitivity peaks are related to overshoot and damping ratio.

\[ M_S = \max_{\omega} |S(j\omega)| \]
\[ M_T = \max_{\omega} |T(j\omega)| \]

\[ S = \rho e^{j\theta} = (1+L)^{-1} \]
\[ \Rightarrow L = -1 + \rho^{-1} e^{-j\theta} \]

Constant \(|S|\) proscribes circle

\[ S^{-1} = 1 + L \]
A Fundamental Tradeoff

Note that $\left[ 1 + L \right]^{-1} + \left[ 1 + L \right]^{-1} L = 1 \Rightarrow S + T = 1$

Making $S$ small improves tracking & disturbance rejection but makes system susceptible to noise $|S|_0 \Rightarrow |T|_1$:

$$E(s) = S(s)R(s) - S(s)G(s)D_1(s) + T(s)D_2(s)$$

Typical design specifications

$$\left| S(j\omega) \right| << 1 \quad \omega \in [0, \omega_1]$$

$$\left| T(j\omega) \right| << 1 \quad \omega \in [\omega_2, \infty]$$

And, there are other limitations.
Bode Waterbed Formula

systems with relative degree 2 or greater:
(Waterbed effect)

\[
\int_{0}^{\infty} \ln |S(j\omega)| \, d\omega = \pi \sum_{\text{ORHP poles}} p_i
\]

\[
\int_{0}^{\infty} \ln |T(j\omega)| \frac{d\omega}{\omega^2} = \pi \sum_{\text{ORHP zeros}} \frac{1}{q_i}
\]

\[\ln |S(j\omega)| \text{ vs. } \omega \text{ for Osprey}\]
Implication of Waterbed Formula

\[ \ln|S(j\omega)| \]

- pops up here
- push down here (e.g., by increasing gain)
V-22 with PID

\[ G(s) = \frac{s^2 + 1.5s + 0.5}{(0.5s + 1)(10s + 1)(20s + 1)} \]
Example: Osprey

Sensitivity function plots for $K=280, 500, 1000$
Example: Osprey

Increased gain yields higher bandwidth (and reduced settling time) but higher sensitivity peak (lower damping ratio and more overshoot).
Cauchy Theorem

**Theorem (Cauchy):** Let $C$ be a simple closed curve in the $s$-plane. $F(s)$ is a rational function, having neither poles nor zeros on $C$. If $C_1$ is the image of $C$ under the map $F(s)$, then

$$Z = P - N$$

where

- $N$ the number of **counterclockwise** encirclements of the origin by $C_1$ as $s$ traverses $C$ once in the **clockwise** direction.
- $Z$ the number of zeros of $F(s)$ enclosed by $C$, counting multiplicities.
- $P$ the number of poles of $F(s)$ enclosed by $C$, counting multiplicities.
Nyquist

- Take $F(s)=1+L(s)$ (return difference, $F=S^{-1}$)
- Choose a $C$ that encloses the entire RHP
- Map into $L$-plane instead of $F$-plane (shift by -1)
Nyquist Theorem

Theorem (Nyquist): If the plot of $L(s)$ (i.e., the image of the Nyquist contour in the $L$-plane) encircles the point $-1 + j0$ in the counterclockwise direction as many times as there are unstable open loop poles (poles of $L(s)$ within the Nyquist contour) then the feedback system has no poles in the RHP.

\[ Z = P - N \]

closed loop poles in RHP = open loop poles in RHP - cc encirclements of -1
Example 1

$$L(s) = \frac{1}{(s + p_1)(s^2 + 2\zeta\omega_0 s + \omega_0^2)}$$

I:  $s = j\omega$ $\Rightarrow$ $L(j\omega) = \frac{1}{(s + p_1)(s^2 + 2\zeta\omega_0 s + \omega_0^2)}$, $0 < \omega < \infty$

II:  $s = \rho e^{j\theta}$, $\rho \to \infty$, $= \frac{\pi}{2} \downarrow -\frac{\pi}{2}$

$$L(\rho e^{j\theta}) = \frac{1}{(\rho e^{j\theta} + p_1)(\rho^2 e^{j2\theta} + 2\zeta\omega\rho e^{j\theta} + \omega^2)} \xrightarrow{\rho \to \infty} 0$$

III:  $s = -j\omega$, III $\to$ I*
Example 2

\[
L(s) = \frac{1}{s(s^2 + 2\zeta\omega_0 s + \omega_0^2)}
\]

I: \( s = j\omega \Rightarrow L(j\omega) = \frac{1}{j\omega(-\omega^2 + j2\zeta\omega_0\omega + \omega_0^2)}, \quad 0 < \omega < \infty \)

II: \( s = \rho e^{j\theta}, \quad \rho \to \infty, \quad \theta = \frac{\pi}{2} \downarrow \frac{-\pi}{2} \)

\[
L(\rho e^{j\theta}) = \frac{1}{\rho e^{j\theta}(\rho^2 e^{j2\theta} + 2\zeta\omega_0 \rho e^{j\theta} + \omega_0^2)} \quad \rho \to \infty \to 0
\]

III: \( s = -j\omega, \quad \text{III} \to \text{I}^* \)

IV: \( s = \varepsilon e^{j\theta}, \quad \varepsilon \to \infty, \quad \theta = \frac{-\pi}{2} \uparrow \frac{\pi}{2} \)

\[
L(\varepsilon e^{j\theta}) = \frac{1}{\varepsilon e^{j\theta}(\varepsilon^2 e^{j2\theta} + 2\zeta\omega_0 \varepsilon e^{j\theta} + \omega_0^2)} \quad \varepsilon \to 0 \to \frac{1}{\varepsilon\omega_0^2} e^{-j\theta}
\]
Example 3

The principle part of the Nyquist plot uses the same data as the bode plot.
Example 4

\[ G(s) = \frac{0.2 \ (s + 1) \ (s^2 + 12)}{s \ (s^2 + 2 \ s + 2) \ (s^2 + 2 \ s + 4)} \]
Example 5

\[ G(s) = \frac{1}{s(s^2 + 2(0.1)s + 1)} \]

\[ G = \frac{1}{s^2 + 2s + 1} \]

\[ G = \frac{1}{(s+0.01)(s^2+2s+1)} \]

\[ G = \frac{1}{s^2 + 2s + 1} \]
Gain & Phase Margin

Assumption: the nominal system is stable.
Example 6

\[ G(s) = \frac{(s + 0.5)}{(s + 1)(s + 10)(s^2 + 0.1s + 1)} \]
V-22 with PID

\[ G(s) = \frac{s^2 + 1.5s + 0.5}{(0.5s + 1)(s + 10)(20s + 1)} \]
V-22 with PID

Complementary sensitivity

sensitivity
V-22 with PID

\[ G(s) = \frac{280 \left( s^2 + 1.5 s + 0.5 \right)}{(0.5 s + 1) s (10 s + 1) (20 s + 1)} \]
V-22 with PID

```matlab
>> s=tf('s');
>> G=280*((s^2+1.5*s+0.5)/s)/(20*s+1)*(10*s+1)*(0.5*s+1));
>> margin(G)
```

Notice that gain margin is negative because system loses stability when gain drops.
V-22 PID+Lead

\[
G(s) = \frac{s^2 + 1.5s + 0.5}{(0.5s + 1)(s + 1)(20s + 1)}
\]
V-22 PID+Lead

Complementary sensitivity

Notice that sensitivity peak is reduced from 20 db to 10 db

Disturbance to output

Command to control

$K=150$
V-22 PID+Lead

\[ G(s) = \frac{150 (s^2 + 1.5s + 0.5)}{(0.5s + 1)s(s + 1)(20s + 1)} \]
V-22 PID+Lead

Gain margin is infinite because the system is never unstable for any $K>0$. 
Summary

• Need to consider 2-4 transfer functions to fully evaluate performance
• Bandwidth is inversely related to settling time
• Sensitivity function peak is related to overshoot and inversely to damping ratio
• Gain and phase margins can be determined from Nyquist or Bode plots
• Sensitivity peak is inversely related to stability margin
• Design tools:
  ▪ Root locus helps place poles
  ▪ Bode and/or Nyquist helps establish robustness (margins) & performance (sensitivity peaks)