MEM 355 Performance Enhancement of Dynamical Systems

The Ultimate State Error

Harry G. Kwatny
Department of Mechanical Engineering & Mechanics
Drexel University
Outline

• Design Criteria – two step process
  ▪ Ultimate state error (this lecture)
  ▪ Transient response (following lectures)
• Ultimate Error Response to Command ~ Unity Feedback Systems
  ▪ Tables ~ Franklin p. 235, Nise p. 383
• Ultimate Error Response to Command ~ General Case
• Example
• Error Response to Disturbance
Control System Performance

• Command tracking
  ▪ Match output to changing command (minimize/eliminate error)
  ▪ Shape transient response (overshoot, damping)

• Disturbance rejection
  ▪ Minimize/eliminate output error while system is subjected to disturbance
  ▪ Shape transient response (overshoot, damping)

• Stabilization
  ▪ Insure adequate degree of stability (decay rate, damping, pole locations)

• Handling qualities
  ▪ From perspective of operator (pilot, driver) – maneuverability, stability

• Reliability, Safety – emerging theory
Unity Feedback Systems

\[ G(s) \triangleq G_p(s)G_c(s) \] open loop transfer function

\[ Y(s) = \frac{G(s)}{1+G(s)} \bar{Y}(s) \] output response transfer function (closed loop)

\[ E(s) = \frac{1}{1+G(s)} \bar{Y}(s) \] error response transfer function (closed loop)

\[ e(t) = \mathcal{L}^{-1}[E(s)], \text{ then plug in } t = \infty \text{ to get } e(\infty) \]

or, much easier, use Final Value Theorem: \[ e(\infty) = \lim_{s \to 0} sE(s) \]
Error Response to Polynomials

Test inputs: polynomial in \( t \), step, ramp, parabola, ...

**step:** \( u(t) \to 1/s \Rightarrow e(\infty) = \lim_{s\to0} s \left( \frac{1}{1 + G(s)} \right) \frac{1}{s} = \frac{1}{1 + \lim_{s\to0} G(s)} \)

**ramp:** \( tu(t) \to 1/s^2 \Rightarrow e(\infty) = \lim_{s\to0} s \left( \frac{1}{1 + G(s)} \right) \frac{1}{s^2} = \frac{1}{\lim_{s\to0} s G(s)} \)

**parab:** \( \frac{1}{2} t^2 u(t) \to 1/s^3 \Rightarrow e(\infty) = \lim_{s\to0} s \left( \frac{1}{1 + G(s)} \right) \frac{1}{s^3} = \frac{1}{\lim_{s\to0} s^2 G(s)} \)

**Position constant:** \( k_p \triangleq \lim_{s\to0} G(s) \)

**Velocity constant:** \( k_v \triangleq \lim_{s\to0} s G(s) \)

**Acceleration constant:** \( k_a \triangleq \lim_{s\to0} s^2 G(s) \)
Transfer Function “Type”

A transfer function $G(s)$ of the form

$$G(s) = K \frac{n(s)}{s^k d(s)}$$

where $s$ is not a factor of $d(s)$ or $n(s)$

is said to be of type $k$.

Note that:

$$k_p = \lim_{s \to 0} G(s) = \begin{cases} c \neq 0 & k = 0 \\ \infty & k \geq 1 \end{cases}$$

$$k_v = \lim_{s \to 0} s G(s) = \begin{cases} 0 & k = 0 \\ \infty & k \geq 2 \end{cases}$$

$$k_a = \lim_{s \to 0} s^2 G(s) = \begin{cases} 0 & k = 0, 1 \\ c \neq 0 & k = 2 \\ \infty & k \geq 3 \end{cases}$$
## Ultimate Error Table ~ Unity Feedback Systems

<table>
<thead>
<tr>
<th>Type</th>
<th>Input 0</th>
<th>Input 1</th>
<th>Input 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u(t)$</td>
<td>$0$</td>
<td>$1$</td>
<td>$2$</td>
</tr>
<tr>
<td>$tu(t)$</td>
<td>$\infty$</td>
<td>$\frac{1}{k_v}$</td>
<td>$0$</td>
</tr>
<tr>
<td>$t^2u(t)/2$</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>$\frac{1}{k_a}$</td>
</tr>
</tbody>
</table>
Example

Consider a unity feedback, type 1 system with

\[ G(s) = \frac{1}{4} \left( \frac{s + 9}{s} \right) \left( \frac{1}{s + 1} \right) \]

Error responses for standard step, ramp, parabola
The closed loop input response transfer function is

\[ G_y(s) = \frac{G}{1 + GH} \]

The error response transfer function is (recall \( e = \bar{y} - y \))

\[ G_e(s) = 1 - G_y(s) = \frac{1 + GH - G}{1 + GH} \]

\[ e(\infty) = \lim_{s \to 0} sG_e(s)\bar{Y}(s) \]
# Ultimate Error Table ~ General Case

<table>
<thead>
<tr>
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<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u(t)$</td>
<td></td>
<td>$\lim_{s \to 0} G_e(s)$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$tu(t)$</td>
<td></td>
<td>$\infty$</td>
<td>$\lim_{s \to 0} sG_e(s)$</td>
<td>0</td>
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<td>$\infty$</td>
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</tr>
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</table>
Each power plant that participates in load tracking receives a command of similar shape.
Example, cont’d

\[ k_i, i = 1, 2 \ldots, n \quad \text{participation factors} \]

\[ k_1 + k_2 + \cdots + k_n = 1 \]
Example: cont’d

“Type” can be useful to improve the tracking of command signals.

\[ y \rightarrow \frac{15s + 0.1}{s} \rightarrow \frac{1}{200s + 1} \rightarrow y \]

Simple power plant model

Step Response

Error response to typical Load command

Note: On a 3000 MW system we have an error of ~ 25 MW.
Example: cont’d

Outer Loop Compensator:

Type 1: \[ G_c(s) = \frac{200s^2 + 16s + 1.5}{s} \]

Type 2: \[ G_c(s) = \frac{200s^2 + 16s + 1.5}{s^2} \]

Error responses with and without outer loop
Ultimate Error Due to Disturbance

Transfer Function $d \rightarrow e : G_{ed} = \frac{G_2}{1 + G_1G_2H}$

$$e(\infty) = \lim_{s \to 0} sG_{ed}(s)D(s)$$
Example: Step Disturbance

\[ e(\infty) = \lim_{s \to 0} sG_{ed}(s)D(s) \]

step disturbance: \( D(s) = 1/s \)

\[ e(\infty) = \lim_{s \to 0} \frac{sG_2}{1 + G_1G_2H} \frac{1}{s} = \frac{1}{\lim_{s \to 0} \frac{1}{G_2(s)} + \lim_{s \to 0} G_1(s)H(s)} \]

To reduce error:

increase type or gain of \( G_1H \)
reduce gain of \( G_2 \)
Example

\[ G_2(s) = \frac{1}{s(s + 25)}, \quad H(s) = 1 \]

\[ G_1(s) = \begin{cases} 
(a): P & \frac{1000}{s} \\
(b): PI & 1000 \frac{(s + 2)}{s} 
\end{cases} \]

(a) \[ G_{\bar{y}e} = \frac{s(s + 25)}{s^2 + 25s + 1000}, \quad G_{de} = \frac{1}{s^2 + 25s + 1000} \]

\[ e_y(\infty) = \lim_{s \to 0} s \frac{s(s + 25)}{s^2 + 25s + 1000} \frac{1}{s} = 0, \quad e_d(\infty) = \lim_{s \to 0} s \frac{1}{s^2 + 25s + 1000} \frac{1}{s} = \frac{1}{1000} \]

(b) \[ G_{\bar{y}e} = \frac{s^2(s + 25)}{s^3 + 25s^2 + 1000s + 2000}, \quad G_{de} = \frac{s}{s^3 + 25s^2 + 1000s + 2000} \]

\[ e_y(\infty) = \lim_{s \to 0} s^2 \frac{s^2(s + 25)}{s^3 + 25s^2 + 1000s + 2000} \frac{1}{s} = 0, \quad e_d(\infty) = \lim_{s \to 0} s \frac{s}{s^3 + 25s^2 + 1000s + 2000} \frac{1}{s} = 0 \]

First confirm that closed loop is stable.
Summary

• Concept of “ultimate error, ”
• Formulae for unity feedback systems in response to commands,
• Introduction of “error constants,”
• The concept of system “type” and its role in command tracking,
• Formulae for general feedback systems in response to commands and disturbances.