MEM 355
Homework 3 Solutions

Problem 1

The given equation is simplified in order to see the pole locations.

\[
\frac{1}{s(s^2 + 3s + 10)} = \frac{1}{s(s + 1.5 + j2.784)(s + 1.5 - j2.784)}
\]

Using this equation, the root locus of the system can be drawn as shown in the steps below.

Step 1

Identify the locations of poles and plot them.

Step 2

Number of branches of root locus.

\[
P - Z \quad 3 - 0 = 3
\]

Root locus has three branches.

Step 3

Parts of real axis which form a part of the root locus.
Entire left half of real axis forms a part of root locus. For reasons please refer to problem 2 solution.

Step 4
Number of asymptotes

Number of asymptotes = \( P - Z \)
\[
\begin{align*}
\text{Number of asymptotes} & = 3 - 0 \\quad \Rightarrow \\quad \text{Number of asymptotes} = 3 \\
\end{align*}
\]

Step 5
Centroid of asymptotes

\[
\sigma_a = \frac{\sum \text{poles} - \sum \text{zeroes}}{P - Z}
\]
\[
\begin{align*}
\sigma_a & = \frac{(0 - 1.5 - j2.784 - 1.5 + j2.784) - (0)}{3} \\
& = -1
\end{align*}
\]

Step 6
Angles of asymptotes

\[
\phi_q = \frac{(2q + 1) \cdot 180}{P - Z}
\]
\[
\begin{align*}
\phi_0 & = \frac{(2 \cdot 0 + 1) \cdot 180}{3} = \frac{(2 \cdot 0 + 1) \cdot 180}{3} = 60^\circ \\
\phi_1 & = \frac{(2 \cdot 1 + 1) \cdot 180}{3} = 180^\circ \\
\phi_2 & = \frac{(2 \cdot 2 + 1) \cdot 180}{3} = 300^\circ
\end{align*}
\]
Step 7

Break away point

Break away points are not applicable in this case as there is no case of a root locus starting at two roots and progressing towards each other. There is a root at 0 and the root locus starting at it is progressing towards $-\infty$. The other two roots are at $-1.5 + j2.784$ and $-1.5 - j2.784$. The root locus starting at these locations proceed along the directions of the asymptotes which are at $60^\circ$ and $300^\circ$.

So the root locus plot is as shown in the figure below.

Step 8

Intersection with imaginary axis.

$$1 + K \frac{1}{s \left(s^2 + 3s + 10\right)} = 0$$

$$s^3 + 3s^2 + 10s + K = 0$$

Substituting $s = j\omega$ in the above equation, we get

$$-j\omega^3 - 3\omega^2 + 10j\omega + K = 0 \quad (11)$$

Equating imaginary parts of Eq. (11) to 0 we get
\[-j\omega^3 + 10j\omega = 0\]
\[10\omega - \omega^3 = 0\]
\[\omega^2 = 10\]
or
\[\omega = 0\]

Equating the real parts of Eq. (11) to 0 and using the value of \(\omega^2 = 10\), we get
\[-3\omega^2 + K = 0\]
\[K = 3\omega^2\]

Using the value \(\omega^2 = 10\) in above, we get
\[K = 30\]

Therefore it can be said that the system is stable for the range \(0 < K < 30\).

MATLAB plot of root locus is as shown below. The value of gain shown in plot is not at \(K=30\), but it shows that the correct value of \(K\) at imaginary axis intersection is between 25 and 47.
Problem 2

The system is described by the equation \( \frac{s+3}{s^3(s+4)} \). Procedure of drawing root locus is similar to that shown for problem 3.

Step 1
Identify the locations of poles and plot them.

There are four poles, three of them at location 0 and one at -4. There is one zero at -3.

Step 2
Number of branches of root locus.

\[
P - Z = 4 - 1 = 3
\]

Root locus has three branches.

Step 3
Parts of real axis which form a part of the root locus.

For reasons please refer to problem 2 solution.
Step 4
Number of asymptotes

Number of asymptotes = \( P - Z \)

\[
\begin{align*}
\text{Number of asymptotes} & = 4 - 1 \\
& = 3 
\end{align*}
\]

Step 5
Centroid of asymptotes

\[
\sigma_a = \frac{\sum \text{poles} - \sum \text{zeros}}{P - Z}
\]

\[
\begin{align*}
\sigma_a & = \frac{(0 - 0 - 0 - 4) - (-3)}{3} \\
& = -\frac{1}{3}
\end{align*}
\]

Step 6
Angles of asymptotes

\[
\phi_j = \frac{(2q + 1) \cdot 180}{P - Z}
\]

\[
\begin{align*}
\phi_0 & = \frac{(2 \cdot 0 + 1) \cdot 180}{3} = \frac{(2.0 + 1) \cdot 180}{3} = 60^\circ \\
\phi_1 & = 180^\circ \\
\phi_2 & = 300^\circ
\end{align*}
\]

Step 7
Break away point
Break away points are not applicable in this case as there is no case of a root locus starting at two roots and progressing towards each other. There are three roots at $\sigma = 0$ and one at $\sigma = -4$. The root starting at -4 proceeds towards $-\infty$. One of the locus starting at $\sigma = 0$ moves towards the zero at $\sigma = -3$. The two other locus starting at $\sigma = 0$ proceed parallel to the asymptotes. The root locus plot is as shown below.

Step 8

There is no positive value of K for which this system can be stable.

Step 9

MATLAB verification is done and the graph obtained is as shown below. It is seen that even for a very small gain, the pole is on the right half of the S plane.
Problem 3

\[ \frac{s + 2}{s(s - 1)(s + 6)^2} \]

Step 1
Identify the locations of poles and plot them.

There are poles at +1, 0 and two poles at -6. There is one zero at -2.

Step 2
Number of branches of root locus.

\[ P - Z = 4 - 1 = 3 \]

Root locus has three branches.

Step 3
Parts of real axis which form a part of the root locus.

Remember that there are two poles at -6, therefore the root locus continues from -6 towards \(-\infty\).

Step 4
Number of asymptotes
Number of asymptotes = \( P - Z \)
\[
= 4 - 1 \\
= 3
\]

Step 5
Centroid of asymptotes
\[
\sigma_a = \frac{\sum \text{poles} - \sum \text{zeroes}}{P - Z}
\]
\[
\sigma_a = \frac{(0 + 1 - 6 - 6) - (-2)}{3} \\
= -\frac{9}{3} = -3
\]

Step 6
Angles of asymptotes
\[
\phi_d = \frac{(2q+1)\times180}{P - Z}
\]
\[
\phi_d = \frac{(2q+1)\times180}{3} \\
\quad \underset{k=0}{=} \frac{(2.0+1)\times180}{3} = 60^\circ
\]
\[
\phi_d = \frac{(2q+1)\times180}{3} \\
\quad \underset{k=1}{=} \quad 180^\circ
\]
\[
\phi_d = \frac{(2q+1)\times180}{3} \\
\quad \underset{k=2}{=} \quad 300^\circ
\]

Step 7
Break away point
There is a situation in the present problem where, the root locus starting from $\sigma = 0$ and $\sigma = 1$ are proceeding towards each other. They cannot end on each other, therefore there has to be a point where they break away and move parallel to the asymptotes. The break away point can be computed as shown.

Consider the characteristic equation of the system $1 + GH = 0$. In this case the characteristic equation is obtained as

$$1 + K \frac{s + 2}{s(s - 1)(s + 6)} = 0$$

Simplifying this, we get

$$s^4 + 11s^3 + 24s^2 - 36s + K(s + 2) = 0$$

Taking $\frac{dK}{ds}$ in the above equation and equating it to 0, we get

$$\frac{dK}{ds} = 0$$

$$\frac{36s - 48s^2 - 33s^2 - 4s^3}{s + 2} - \frac{36s - 24s^2 - 11s^3 - s^4}{(s + 2)^2} = 0$$

$$\therefore s = 0.488$$

Solving for the values of $s$ we get $s = 0.488$. Since this value lies on the real axis and within the range 0 to 1, this is taken as the breakaway point and other solutions for value of $s$ are discarded. Therefore the complete root-locus is as shown in the figure below.

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Step 8
Since a part of the root locus is on the right half of $s$ plane for all values of $K$, this system cannot be stabilized with the presently used controller. Better controllers need to be considered.

Step 9
MATLAB verification
The root-locus made with MATLAB is as shown below.