INTRODUCTION

The increasing awareness of the engineering community regarding the stochastic nature of structural properties and loads, such as earthquake and wind loads, has led to a renewed interest in simulation techniques. Monte Carlo simulations appear to be the most versatile tool in stochastic engineering mechanics, as, for the vast majority of problems, only the deterministic solution is known or can be evaluated with sufficient accuracy. The emphasis placed on the development of simulation techniques by the research community is illustrated by the following list of publications; it is noted, however, that the list is by no means exhaustive.

One of the techniques that has been widely applied in engineering problems is the spectral representation method. The concept of representations of the Gaussian random processes was introduced by Rice (1944a,b) [reprinted in Rice (1954)], but the use of the approach in generating simulations of random processes and fields originates from Shinozuka (Shinozuka 1971, 1972; Shinozuka and Jan 1972). The methodology has been presented in a comprehensive article by Shinozuka (1987), and was more recently elaborated on in Shinozuka and Deodatis (1991, 1996), and references therein. In its initial formulation, the approach dealt with the summation of large numbers of weighted trigonometric functions, which is, computationally, not efficient. Yang (1972) introduced the fast Fourier transform (FFT) technique in simulating envelopes of random processes, and Shinozuka (1974) extended the approach to random processes and fields. The use of the FFT dramatically reduces the computational requirements for simulations. Improvements on the approach and evaluations of its properties and capabilities have been reported by various researchers over the past few years (Lutes and Wang 1991; Ditlevsen 1992; Zerva 1992; Grigoriu 1993a; Mignolet and Harish 1996).

Simulations based on ARMA models have also been used extensively (Kozin 1988; Ellis and Cakmak 1991; Conte et al. 1992; Mignolet and Spanos 1992; Spanos and Mignolet 1992). Simulations of nonstationary time series by means of processes modulated by time-varying functions (Kuirughiann and Crempien 1989; Yeh and Wen 1990), and by means of random trigonometric functions (Grigoriu 1993b) have also been reported. A hybrid discrete Fourier transform (DFT) and digital filtering approach for multivariate random processes was presented by Li and Kareem (1993, 1997). A simulation approach based on the sampling theorem was proposed by Grigoriu and Balou (1993). Simulations of random fields were generated by Fenton and Vanmarcke (1990) by means of the local average subdivision method, and by Mantoglou and Wilson (1981) with the turning bands method. Simulations based on wavelet transforms were reported by Gurley and Kareem (1994) and Zeldin and Spanos (1996).

The appropriate simulation technique for the particular problem at hand depends on the characteristics of the problem itself (Fenton 1994). Once the technique is selected, a major consideration is the computational effort associated with it; even with the increasing capabilities of computers, this computational effort can be, in cases, prohibitive. The present work presents an efficient approach for the simulation of homogeneous and partially isotropic random fields based on spectral representation. In existing techniques, the property of partial isotropy in a homogeneous random field is not considered, and the field is simulated as homogeneous only. It is shown herein that, by incorporating the partial isotropy of the field in the simulation algorithm, the computational effort required for the simulations is significantly reduced as compared with the case when only the homogeneity in the field is taken into account. The second section of this paper presents a rigorous description of the spectral representation methodology, on which the approach is based, for the simulation of homogeneous processes and fields. Both Rice’s (1944a,b) (Method 1) and Shinozuka’s (1972) (Method 2) simulation techniques are presented and can be utilized with the present approach. Method 1 yields simulations with random amplitudes and phases, whereas Method 2 results in simulations with random phases only. For completeness and clarity in the present formulation, the properties of the two techniques for the simulation of random processes are also presented. Their differences in terms of Gaussianity, ergodicity, and spectral estimation are highlighted; it is noted that Grigoriu (1993a), in a thorough comparison of the two techniques, concluded that they are both valid for simulation purposes. The third section of this paper presents the developed approach for the simulation of homogeneous and partially isotropic random fields; the algorithm for the simulations by means of the FFT is also presented. Recently, Mignolet and Harish (1996) compared the performance of the spectral representation algorithm, the randomized spectral representation scheme (Shinozuka and Jan 1972), and the random frequencies algorithm (Shinozuka 1971) for random processes, and concluded that, irrespective of computational effort, the latter performs generally better in terms of
first- and second-order distributions than the other two. However, because computational efficiency is one of the primary goals of the present work, a fixed location of frequency per interval was selected, so that the FFT could be readily applied. The fourth section illustrates the derivations of the paper with an example regarding simulations of spatially variable seismic ground motions experiencing loss of coherence.

SIMULATION OF HOMOGENEOUS RANDOM FUNCTIONS

Simulation of Stationary Random Processes

Let \( \xi(t), t \in T \) be a second-order zero mean random function defined on some set \( T \), that is we assume \( \text{E}[|\xi(t)|^2] < \infty \) for any \( t \in T \). A general idea for simulating such a function is to use Kahlrun’s theorem (Kahlrun 1946). It states that if the autocovariance \( R_{\xi}(s, t) \) of the function \( \xi(t) \) can be expressed in the form

\[
R_{\xi}(s, t) = \int_{0}^{\infty} \alpha(t, \omega) \mathcal{G}_{\xi}(\omega) d\omega
\]

with \( \omega \) indicating frequency, then this function can be simulated using the formula

\[
\xi(t) = \int_{0}^{\infty} \alpha(t, \omega) dZ(\omega)
\]

where \( dZ(\omega) \) represents an orthogonal increment process satisfying \( \text{E}[dZ(\omega)] = 0 \), and \( \text{E}[dZ(\omega_1) dZ(\omega_2)] = \mathcal{G}_{\xi}(\omega_1) \delta(\omega_1 - \omega_2) d\omega_1 d\omega_2 \). In practical applications, a discrete approximation of (2) is used.

As an example, consider the case when \( T = \mathbb{R} \) and \( \xi(t) \) is a real-valued Gaussian stationary process with autospectral density \( \mathcal{G}_{\xi}(\omega) \). In this case, according to the Bochner-Wiener-Khintchine representation, we have

\[
R_{\xi}(s, t) = \int_{\mathbb{R}} e^{i\omega s} \mathcal{G}_{\xi}(\omega) d\omega
\]

Because \( \xi(t) \) is real-valued process, \( \mathcal{G}_{\xi}(-\omega) = \mathcal{G}_{\xi}(\omega) \) and

\[
R_{\xi}(s, t) = \int_{0}^{\infty} \cos(\omega t) \mathcal{G}_{\xi}(\omega) d\omega
\]

\[
+ \int_{0}^{\infty} \sin(\omega t) \mathcal{G}_{\xi}(\omega) d\omega
\]

where \( \mathcal{G}_{\xi}(\omega) \) is one-sided spectral density of the process \( \xi(t) \). Each of the two terms in (3) satisfies condition (1) of Kahlrun’s theorem. Consequently, we can express the random process as a sum of two independent random processes, each having the form (2), as follows:

\[
\xi(t) = \int_{0}^{\infty} \cos(\omega t) dZ_{\xi}(\omega) + \int_{0}^{\infty} \sin(\omega t) dZ_{\xi}(\omega)
\]

where

\[
\text{E}[Z_{\xi}(A)Z_{\xi}(B)] = \delta_{AB} \int_{0}^{\pi} \mathcal{S}_{\xi}(\omega) d\omega
\]

for any measurable subsets \( A \) and \( B \) of \([0, \infty)\). The integrals on the right-hand side of (4) can be approximated using Riemann integral sums as follows: Let \( \omega_0 \) be the upper cutoff frequency, above which the values of the frequency spectrum are insignificant for practical purposes. We divide the interval \([0, \omega_0]\) into \( N \) equal parts, each having length \( \Delta \omega = \omega_0/N \). Using (5) we see that for any \( 1 \leq k \leq N \) and \( l = 1, 2 \), the random variables \( Z_{\xi}((k - 1)\Delta \omega, k\Delta \omega) \) are independent Gaussian variables with zero mean and variance

\[
\text{E}[Z_{\xi}((k - 1)\Delta \omega, k\Delta \omega)] = \int_{(k-1)\Delta \omega}^{k\Delta \omega} \mathcal{S}_{\xi}(\omega) d\omega \approx \mathcal{S}_{\xi}(\omega_0) \Delta \omega
\]

where \( \omega_0 \) is some point in the interval \([(k - 1)\Delta \omega, k\Delta \omega]\). To use the FFT in the simulations we will assume that \( \omega_0 \) is chosen in such a way that for any \( k \geq 1 \), \( \omega_{k+1} - \omega_k = \Delta \omega \). It follows from the above, that an expression for simulation has the following form of Riemann integral sum:

\[
f(t) = \sum_{k=0}^{N} \sqrt{\mathcal{S}_{\xi}(\omega_0) \Delta \omega} \left[ \cos(\omega_0 t) \eta_{1k} + \sin(\omega_0 t) \eta_{2k} \right]
\]

where \( \eta_{1k} \) and \( \eta_{2k} \) are independent Gaussian random variables with zero mean and unit variance. Herein, \( \xi \) denotes the random function for simulation and \( f \) denotes the result of simulation. This is essentially the method of Rice (1944a,b), and will be referred to hereafter as Method 1. The properties of the function \( f(t) \) in (6) are as follows:

- It is Gaussian.
- It is the sum of trigonometric functions with random amplitudes. Written, alternatively, as

\[
f(t) = \sum_{k=0}^{N} \sqrt{\mathcal{S}_{\xi}(\omega_0) \Delta \omega} \left[ \cos(\omega_0 t) \eta_{1k} + \sin(\omega_0 t) \eta_{2k} \right]
\]

it is the sum of trigonometric functions having random amplitudes and phases.
- It is periodic with period \( T = 2\pi/\Delta \omega \), assuming the ratio \( \omega_0/\Delta \omega \) is equal to a rational number \( p/q \), where \( p \) and \( q \) are mutually prime numbers.
- Its autocorrelation function \( R_{\xi}(\tau) \) is equal to

\[
R_{\xi}(\tau) = \Delta \omega \sum_{k=1}^{N} \mathcal{S}_{\xi}(\omega_0) \cos(\omega_0 \tau); \quad \lim_{\Delta \omega \to 0} R_{\xi}(\tau) = R_{\xi}(\tau)
\]

- It is ergodic of the first order, if \( \omega_0 \neq 0 \), but not ergodic of the first order, if \( \omega_0 = 0 \) and \( S_{\xi}(0) \neq 0 \), because in this case one obtains

\[
\lim_{\Delta \omega \to 0} \frac{1}{T} \int_{0}^{T} f(s) ds = \sqrt{\mathcal{S}_{\xi}(0) \Delta \omega} \eta_{1k} \neq 0
\]

This observation implies that using a rectangular rule for the approximate calculation of the stochastic integral (2) leads to a process \( f(t) \) that is not ergodic of the first order. However, this disadvantage can be overcome using a centroid rule, that is, \( \omega_0 = [(2k - 1)/2] \Delta \omega \). Moreover, it can be shown, that under the above conditions of ergodicity, process (6) is also ergodic over one period.

- It is not ergodic of the second order. It can easily be shown that

\[
\lim_{\Delta \omega \to 0} \frac{1}{T} \int_{0}^{T} f(s) f(s + \tau) ds
\]

\[
= \frac{1}{2} \Delta \omega \sum_{k=1}^{N} \mathcal{S}_{\xi}(\omega_0) \cos(\omega_0 \tau) [\eta_{1k}^2 + \eta_{2k}^2] \neq R_{\xi}(\tau)
\]

However, the process is asymptotically ergodic of the second order, i.e.

\[
\lim_{\Delta \omega \to 0} \frac{1}{T} \int_{0}^{T} f(s) f(s + \tau) ds = R_{\xi}(\tau)
\]

if \( \int_{0}^{\infty} \mathcal{S}_{\xi}(\omega) d\omega < \infty \).
Shinozuka (1972) and Shinozuka and Jan (1972), proposed an alternative approach, that yields simulations ergodic of the second order, which will be denoted hereafter as Method 2

\[
f(t) = \sqrt{2} \sum_{k=1}^{N} \sqrt{S_{\text{det}}(\omega_k)} \cos(\omega_k t + \psi_k) \tag{9}
\]

where \(\psi_k, 1 \leq k \leq N = \text{independent random variables, having uniform distribution on the interval [0, 2\pi]}.\) The properties of (9) are as follows:

- It is not Gaussian, but asymptotically Gaussian as \(N \to \infty\) due to the central limit theorem.
- It is a sum of trigonometric functions with random phases.
- If the ratio \(\omega_0/\Delta \omega\) is a rational number, then \(f(t)\) is a periodic function with the same period as in Method 1.
- The autocorrelation function \(R_x(\tau)\) of \(f(t)\) is as in Method 1.
- It is ergodic of the first order over an infinite time domain, and, also, over one period, under the same conditions as in Method 1, i.e., if \(\omega_0 \neq 0\), or if \(S_{\text{det}}(0) = 0\). As in Method 1, ergodicity of the first order is preserved if the centroid rule is used.
- It is ergodic of the second order, in contrast to Method 1. Indeed, it can easily be shown that

\[
\lim_{t \to + \infty} \frac{1}{t} \int_{0}^{t} f(\tau) d\tau = R_x(\tau)
\]

It can also be easily shown that \(f(t)\) is ergodic of the second order over one period.

It is noted that both simulation formulas (6) and (9) can be implemented computationally in a very efficient manner using shifted FFT (Zerva 1992).

Simulation of Homogeneous Random Fields

Let \(\xi(x, t), x \in \mathbb{R}^d, t \in \mathbb{R}\) be a zero mean Gaussian homogeneous random field. In the general case its spectral density \(G_{\xi}(k, \omega)\), with \(k\) indicating a wave-number vector, has only one symmetry

\[
G_{\xi}(-k, -\omega) = G_{\xi}(k, \omega); (k, \omega) \in \mathbb{R}^{d+1}
\]

Therefore, the calculation of its autocorrelation function can be done using integration over only the half-space \(\{|k, \omega|: \omega \geq 0\}\). Denoting now by \(S_{\xi}(k, \omega)\), the one-sided spectral density, the autocorrelation function \(R_{\xi}(\Delta x, \Delta t) = E[\xi(x_1, t_1)\xi(x_2, t_2)]\), between the points \((x_1, t_1)\) and \((x_2, t_2)\) has the form

\[
R_{\xi}(\Delta x, \Delta t) = \int_{0}^{\infty} \int_{0}^{\infty} \cos(k, \Delta x_1 + \cdots + k, \Delta x_d + \omega \Delta t) \cdot S_{\xi}(k, \omega) d\omega dk
\]

Here \(\Delta x = x_1 - x_2, \Delta t = t_1 - t_2\). We select an upper cutoff frequency \(\omega_0\), upper cutoff wave numbers \(k_1, \ldots, k_M\), and integers \(N, N_1, \ldots, N_d\) and define \(\Delta \omega = \omega_0/N, \Delta k = k_0/N_1, 1 \leq M \leq M_t\). To preserve ergodicity of the first order, we select to use the centroid rule: \(\omega_0 = (2k - 1)\Delta \omega/2\), and \(\Delta k = (-2N_1 + 2k_1 - 1)\Delta k/2\).

Simulations based on Method 1 are generated from

\[
f(x, t) = \sqrt{2} \sum_{k=1}^{M_t} \sum_{l=1}^{N} \sum_{t_l=1}^{N_l} \sqrt{S_{\text{det}}(k_{1l}, \ldots, k_{dl}, \omega_0)} \times [\cos(k_{1l}x_1 + \cdots + k_{dl}x_d + \omega_0 t_l)\eta_{l_1 \ldots l_d}^t + \sin(k_{1l}x_1 + \cdots + k_{dl}x_d + \omega_0 t_l)\eta_{l_1 \ldots l_d}^s] \tag{10}
\]

where \(\eta_{l_1 \ldots l_d}^t, m = 1, 2\) are independent standard Gaussian random variables.

Simulations based on Method 2 are generated according to (Shinozuka 1987)

\[
f(x, t) = \sqrt{2} \Delta \omega \Delta k_1, \ldots, \Delta k_d \sum_{k=1}^{N} \sum_{l=1}^{N_1} \cdots \sum_{t_l=1}^{N_d} \sqrt{S_{\text{det}}(k_{1l}, \ldots, k_{dl}, \omega_0)} \times \cos(k_{1l}x_1 + \cdots + k_{dl}x_d + \omega_0 t_l + \psi_{l_1 \ldots l_d}) \tag{11}
\]

where \(\psi_{l_1 \ldots l_d}^t\) sequence of independent random variables having uniform distribution on the interval [0, 2\pi].

The shifted FFT (Zerva 1992) can be readily applied to both simulation formulas \((10)\) \& \((11)\). It is noted that for both methods, simulation over a grid of points in the space-time domain, assuming \(N\) points in each direction, requires a computational effort proportional to \(N^2M_t^2\) if no FFT is used. The use of FFT reduces this computational effort drastically to a quantity proportional to \((N \log N)^{M_t+1}\).

**HOMOGENEOUS AND PARTIALLY ISOTROPIC RANDOM FIELDS**

**Mathematical Background**

Let \(\xi(x, t)\) be a zero mean Gaussian homogeneous random field in the space-time domain \(\mathbb{R}^{d+1}\). Herein \(x \in \mathbb{R}^d\) denotes a point in the space domain, and \(t \in \mathbb{R}\) a point in the time domain. Let \(g\) be an arbitrary rotation in the space domain, which is a linear operator in \(\mathbb{R}^d\) satisfying the conditions \(g^* = g^{-1}\), and \(det g = 1\). The random field \(\xi(x, t)\) is isotropic with respect to the space variable, if its autocorrelation function \(R_x(\Delta x, \Delta t)\) satisfies the condition \(R_x(g \Delta x, \Delta t) = R_x(\Delta x, \Delta t)\) for any rotation \(g\).

Let \(S_{\xi}(k, \omega)\) be the one-sided spectral density of the random field \(\xi(x, t)\), defined on the wave-number frequency domain. The Bochner-Wiener-Khintchine theorem states

\[
R_x(\Delta x, \Delta t) = \int_{0}^{\infty} \int_{0}^{\infty} e^{i(k, \Delta x) + i\omega \Delta t} S_{\xi}(k, \omega) dk d\omega \tag{12}
\]

where \((k, \Delta x) = k_1\Delta x_1 + \cdots + k_d\Delta x_d\). Assuming that the random field is partially isotropic, i.e., isotropic with respect to the space variables, but not with respect to time, it is easy to show that \(S_{\xi}(gk, \omega) = S_{\xi}(k, \omega)\) and, consequently, one can write \(S_{\xi}(k, \omega) = S_{\xi}(k, \omega, k)\), where \(k = |k|\). Using Formulas 3.3.2.3 and 2.5.40.6 in (Prudnikov et al. 1988), we can rewrite (12) as

\[
R_x(\Delta x, \Delta t) = (2\pi)^{2d} \int_{0}^{2\pi} \int_{0}^{2\pi} J_{\text{nl}}(\Delta \omega) \cos(\omega \Delta t) S_{\xi}(k, \omega) dk d\omega \tag{13}
\]

where \(J_{\text{nl}}\) is Bessel function of the \(n\)th order. It must be noted that Kahlrunen’s theorem cannot be applied to (13), because (13) does not have form (1). To obtain such a form, we recall that \(\Delta x = x_1 - x_2\), and \(\Delta t = t_1 - t_2\), where \((x_1, t_1)\) and \((x_2, t_2)\) are the points between which we evaluate the autocorrelation function, and use the general addition theorem for Bessel functions (Ivanov and Leonenko 1989, p. 14); then (13) becomes

\[
R_x(\xi_1, t_1; \xi_2, t_2) = (2\pi)^{d+1} \sum_{n=0}^{\infty} \sum_{M=0}^{\infty} \int_{0}^{2\pi} \int_{0}^{2\pi} \int_{0}^{2\pi} S_n^m(\xi_1) S_n^m(\xi_2) \times J_{\text{nl}}(\Delta \omega) J_{\text{nl}}(\Delta \omega) \cos(\omega \Delta t) \sin(\omega \Delta t) S_{\xi}(k, \omega) dk d\omega \tag{14}
\]
where \( S_n \) = spherical harmonics of \( m \)-th order in the \( M \)-dimensional space; and \( h(M, m) = (2m + M - 2) ![M + m - 3] / ![M - 2] ![m] \) = number of these harmonics. Note that (14) has the form (1) required by Karhunen’s theorem. To illustrate the approach, we next consider the case when the case dimension is \( M = 2 \), i.e., a 2D space domain.

Let \( (r, \varphi) \) be the polar coordinates in the space domain. In this case, we have \( h(2, m) = 2, m \geq 1, h(2, 0) = 1 \), and the spherical harmonics have the form (Ivanov and Leonenko 1989)

\[
S_1^1(\varphi) = \frac{1}{\sqrt{2\pi}} e^{-i\varphi}, \quad m \geq 0; \quad S_2^0(\varphi) = \frac{1}{\sqrt{2\pi}} e^{-i\varphi}, \quad m \geq 1
\]

Substituting these expressions in (14), and making use of the Bessel functions’ symmetry property (Watson 1966), namely, \( I_{-n}(z) = (-1)^n I_n(z) \), we obtain the desired expression for the autocorrelation function, written in a form consistent with Karhunen’s theorem, as

\[
R_{ik}(r_1, \varphi_1; r_2, \varphi_2, t_2) = 2\pi \sum_{l=0}^{\infty} \cos(l\varphi_1)\cos(l\varphi_2) + \sin(l\varphi_1)\sin(l\varphi_2) \times \int_0^\infty \int_0^\infty I_l(kr_1)I_{m}^{*}(kr_2) \cos(\omega t_1)\cos(\omega t_2) + \sin(\omega t_1)\sin(\omega t_2) d\omega k dk \(15\)
\]

The sum in (15) can be truncated to include a relatively small number of terms, as will be shown subsequently in the example application. It is also noted that (15) involves only 2D integrations.

### Simulation of Homogeneous and Partially Isotropic Random Fields

Let \( \xi(x, t) \) be a homogeneous random field in the space-time domain \( \mathbb{R}^{M+1} \), which is isotropic with respect to the space variables. One can use directly the simulation approach of Method 1 ([10]) or Method 2 ([11]), described in the section entitled Simulation of Homogeneous Fields, for homogeneous-only random fields. One must note, however, that simulations based on (10) or (11) require \( (M + 1) \)-fold Fourier transforms and the storage of \( (M + 1) \)-dimensional arrays in memory. In what follows we will present a simulation technique that takes advantage of the presence of isotropy. For simplicity we will consider, again, the case \( M = 2 \), i.e., the random field is 2D in space, in addition to its time dimension.

#### Method 1

Using Method 1 to simulate the random field for \( M = 2 \), we obtain from (15)

\[
f(r, \varphi, t) = 2\pi \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \sqrt{k_1} S_k(k_1, \varphi, \omega) J_l(k_1 r) \times \left[ \cos(l\varphi)\cos(\omega t) \eta_{l,1} + \sin(\omega t)\sin(l\varphi) \eta_{l,2} \right] + \sin(\omega t)\cos(\omega t) \eta_{l,1} + \sin(l\varphi)\sin(\omega t) \eta_{l,2}
\]

where \( \eta_{m,n} \), \( 1 \leq m \leq 4 \) = independent standard Gaussian random variables. This random field has the following properties:

- It is Gaussian.
- Its autocorrelation function is equal to

\[
R_{ij}(r_1, \varphi_1; r_2, \varphi_2, t_2) = \sum_{k=1}^{N} C_k(r_1, \varphi_1; r_2, \varphi_2) \cos(\omega k(t_1 - t_2))
\]

where

\[
C_k(r_1, \varphi_1; r_2, \varphi_2) = 2\pi \Delta \omega \sum_{l=0}^{N} k_1 S_k(k_1, \varphi_1, \omega_1) \times \sum_{m=1}^{N} \cos(l(\varphi_1 - \varphi_2)) J_l(k_1, \varphi_1) J_l(k_1, \varphi_2)
\]

We also have

\[
\lim_{T \rightarrow \infty} P_{\rho,\Delta \omega} = R_{di}(p, t_1 - t_2) = R_{di}(p, t_1 - t_2)
\]

where \( \rho = \text{distance between points } x_1 \) and \( x_2 \).

This random field is isotropic because its autocorrelation function depends only on the difference \( |\varphi_1 - \varphi_2| \).

It is nonhomogeneous, but asymptotically homogeneous, as \( L \rightarrow \infty \), due to Neumann’s addition theorem (Watson 1966).

- The period of any time history \( f(r, \varphi, t) \) is equal to \( T = 2\pi q/\Delta \omega \), assuming the ratio \( \omega_0/\Delta \omega \) is equal to a rational number \( \rho \) \( p/q \), where \( p \) and \( q \) are mutually prime numbers.

- It can be easily shown that the time histories \( f(r, \varphi, t) \) are ergodic of the first order under similar conditions as in the section entitled Simulation of Stationary Random Processes, namely, if \( \omega_0 \neq 0 \) or \( S_0(k_1, 0) \neq 0 \) for any \( k_1 \).

- The time histories \( f(r, \varphi, t) \) are only asymptotically ergodic of the second order. To show this, we rewrite (16) as

\[
f(r_1, \varphi_1, t) = \sum_{k=1}^{N} A_k(r_1, \varphi_1) \cos(\omega t + B_k(r_1, \varphi_1) \sin(\omega t))
\]

where

\[
A_k(r_1, \varphi_1) = 2\pi \Delta \omega \sum_{l=1}^{N} \sum_{k=1}^{N} \sqrt{k_1} S_k(k_1, \varphi_1, \omega) J_l(k_1 r_1) \times \left[ \cos(l\varphi_1) \eta_{l,1} + \sin(l\varphi_1) \eta_{l,2} \right]
\]

and

\[
B_k(r_1, \varphi_1) = 2\pi \Delta \omega \sum_{l=1}^{N} \sum_{k=1}^{N} \sqrt{k_1} S_k(k_1, \varphi_1, \omega) J_l(k_1 r_1) \times \left[ \cos(l\varphi_1) \eta_{l,1} + \sin(l\varphi_1) \eta_{l,2} \right]
\]

For two fixed locations \( (r_1, \varphi_1) \) and \( (r_2, \varphi_2) \) we define the random process

\[
Y_{\Delta \omega, \Delta t}(\tau) = \lim_{T \rightarrow \infty} \frac{1}{T - t} \int_{0}^{T} f(r_1, \varphi_1, s) f(r_2, \varphi_2, s + \tau) ds
\]

Calculating the limit of (18), we obtain

\[
Y_{\Delta \omega, \Delta t}(\tau) = \frac{1}{\Delta \omega} \sum_{\omega} \left[ A_1(r_1, \varphi_1) A_1(r_2, \varphi_2) + B_1(r_1, \varphi_1) B_1(r_2, \varphi_2) \cos(\omega \tau) \right] - R_{\rho, \Delta \omega}(r_1, \varphi_1, r_2, \varphi_2, t_1 + t_2)
\]

which proves the lack of ergodicity of the second order. Asymptotical ergodicity of the second order means that the random process \( Y_{\Delta \omega, \Delta t}(\tau) \) has a nonrandom limit equal to the ensemble average \( \left\langle R_{\rho, \Delta \omega}(r_1, \varphi_1, r_2, \varphi_2, t_1 + t_2) \right\rangle \) as \( \Delta \omega \rightarrow 0 \). It is easy to show that

\[
E[A_1(r_1, \varphi_1) A_1(r_2, \varphi_2)] = E[B_1(r_1, \varphi_1) B_1(r_2, \varphi_2)] = C_1(r_1, \varphi_1; r_2, \varphi_2)
\]

where \( C_1(r_1, \varphi_1; r_2, \varphi_2) \) was defined in (17), and consequently

\[
E[Y_{\Delta \omega, \Delta t}(\tau)] = R_{\rho}(r_1, \varphi_1, t_1, r_2, \varphi_2, t_1 + t_2)
\]
The additional condition for asymptotic ergodicity, namely
\[
\lim_{\Delta \omega \to 0} E[(\mathbf{X}_{\Delta \omega}(\tau) - R_0(r_1, \varphi_1, t_1, r_2, \varphi_2, t + \tau))^2] = 0
\]
can be easily verified.

**Method 2**

Using Method 2 to simulate the random field for \( M = 2 \), we obtain from (15)
\[
f(r, \varphi, t) = 2\sqrt{\pi} \Delta \omega \sum_{j=1}^{L} \sum_{l=1}^{N_j} \sum_{k=1}^{N} \sqrt{\kappa_j S_{\eta_j}(\kappa_j, \omega_j) f_j(k_j, r_j)}
\]
\[
\times [\cos(k_1 \gamma_1) \cos(\omega_1 t) + \psi_{j_1} \eta_{j_1} + \sin(k_1 \gamma_1) \cos(\omega_1 t) + \psi_{j_1} \eta_{j_1}]
\]
(20)

where \( \psi_{j_1}, \eta_{j_1}, m = 1, 2 \) are independent random variables having uniform distribution on the interval [0, 2\( \pi \)]. The time histories (20) have all properties mentioned in the simulation of Stationary Random Processes section for 1D processes, simulated by Method 2, except ergodicity of the second order. However, we will show that, if \( f_j(\varphi_j, t) \) is integrated over a certain spatial domain, the time histories become asymptotically ergodic of the second order. To prove this, we consider, as before, the random process (18). Substituting (20) into (18), expanding the right-hand side of (18), and taking expectations on both sides, we obtain
\[
E[Y_{\Delta \omega}(\tau)] = 2\sqrt{\pi} \Delta \omega \sum_{j=1}^{L} \sum_{l=1}^{N_j} \sum_{k=1}^{N} \sqrt{\kappa_j S_{\eta_j}(\kappa_j, \omega_j) f_j(k_j, r_j)}
\]
\[
\times [\cos(k_1 \gamma_1) \cos(\omega_1 t) + \psi_{j_1} \eta_{j_1} + \sin(k_1 \gamma_1) \cos(\omega_1 t) + \psi_{j_1} \eta_{j_1}]
\]
(21)

Using the elementary inequalities \((a + b)^2 \leq 2(a^2 + b^2)\) and \(E[\cos(\omega_1 t) + \psi_{j_1} \eta_{j_1} + \sin(\omega_1 t) + \psi_{j_1} \eta_{j_1}] \leq 1/2\), and the corollary to Neumann’s additional theorem, \( \sum_{\eta_{j_1} = \pm 1} f_j(k_j, r_j) = 1 \), we obtain
\[
E[Y_{\Delta \omega}(\tau) - R_0(r_1, \varphi_1, t_1, r_2, \varphi_2, t + \tau)]^2 \leq 4\pi(\Delta \kappa)^2(\Delta \omega)^2
\]
\[
\cdot \sum_{j=1}^{L} \sum_{l=1}^{N_j} \sum_{k=1}^{N} \kappa_j S_{\eta_j}(\kappa_j, \omega_j) f_j(k_j, r_j) f_j(k_j, r_j)
\]
\[
\rightarrow 4\pi^2 \Delta \omega \sum_{j=1}^{L} \sum_{l=1}^{N_j} \sum_{k=1}^{N} \kappa_j S_{\eta_j}(\kappa_j, \omega_j) f_j(k_j, r_j) f_j(k_j, r_j)
\]
due to Neumann’s addition theorem. The generated time histories are, again, stationary.

**Simulation of Homogeneous and Partially Isotropic Random Fields Using FFT**

Let \((x, y, t)\) be a zero mean Gaussian homogeneous random field, isotropic with respect to the space variables and having one-sided isotropic spectral density \( S_\eta(\kappa, \omega) \). Let \((r, \varphi)\) be the polar coordinates in the \((x, y)\)-plane. We choose an upper cutoff frequency \( \omega_1 \) and an upper cutoff wave number \( \kappa_1 \), above which the contribution of \( S_\eta(\kappa, \omega) \) to the simulations is insignificant for practical purposes. We define the frequency and wave-number nodes based on centroid rule as \( \omega_k = (2k - 1)\Delta \omega/2 \) and \( \kappa_k = (2k_1 - 1)\Delta \kappa/2 \), \( k = 1, \ldots, N \), \( k_1 = 1, \ldots, N_1 \).

For Method 1 the algorithm for the simulation based on (16) and using FFT consists of the following steps:

1. Define two arrays for \( 1 \leq l \leq N_1 \) and \( -L \leq l \leq L \)

\[
A_{ll}^{(i)} = \left\{ \begin{array}{ll}
2\sqrt{2\pi} \Delta \omega \cos(\omega_1 t) \exp(\eta_{ij_1}), & 1 \leq k \leq N_i \\
0, & N + 1 \leq k \leq 2N_i
\end{array} \right.
\]

\[
B_{ll}^{(i)} = \left\{ \begin{array}{ll}
2\sqrt{2\pi} \Delta \omega \cos(\omega_1 t) \exp(\eta_{ij_1}), & 1 \leq k \leq N_i \\
0, & N + 1 \leq k \leq 2N_i
\end{array} \right.
\]

where all \( \eta \) are independent Gaussian random variables with zero mean and unit variance.

2. Calculate and save the values of \( \text{Re}[a_{ij_1}^{(i)}] \) and \( \text{Re}[b_{ij_1}^{(i)}] \), where \( a_{ij_1}^{(i)} \) and \( b_{ij_1}^{(i)} \) are the FFT of the column of the arrays \( A_{ll}^{(i)} \) and \( B_{ll}^{(i)} \) defined in Step 1, and \( c_j = \exp(i\pi(\eta_j - 1)/(2N)) \).

3. Calculate the values of the time histories \( f_j(r, \varphi) = f(r, \varphi, t) \) at any location \((r, \varphi)\) and at discrete times \( t = (j - 1)\Delta t, \Delta t = \pi/\omega_1, j = 1, \ldots, 2N \), using the expression

\[
f_j(r, \varphi) = \sum_{i=1}^{L} \sum_{j=1}^{N_j} f_j(k_j, r_j)(\cos(k_j \gamma_j) \text{Re}[a_{ij_1}^{(i)}] + \sin(k_j \gamma_j) \text{Re}[b_{ij_1}^{(i)}])
\]

(22)

Eq. (22) produces simulations of sampled time histories over half a period \( T/2 = 2\Delta t \). Simulations over the second half period are obtained as \( f_j(r, \varphi) = -f_j(r, \varphi) \). For Method 2 the algorithm for simulation based on (20) and using FFT consists of the same three steps as that for Method 1, except that the arrays \( A_{ll}^{(i)} \), \( B_{ll}^{(i)} \) in Step 2 are defined as follows:

\[
A_{ll}^{(i)} = \left\{ \begin{array}{ll}
4\sqrt{4\pi} \Delta \omega \cos(\omega_1 t) \exp(\eta_{ij_1}), & 1 \leq k \leq N_i \\
0, & N + 1 \leq k \leq 2N_i
\end{array} \right.
\]

\[
B_{ll}^{(i)} = \left\{ \begin{array}{ll}
4\sqrt{4\pi} \Delta \omega \cos(\omega_1 t) \exp(\eta_{ij_1}), & 1 \leq k \leq N_i \\
0, & N + 1 \leq k \leq 2N_i
\end{array} \right.
\]

where all \( \psi \) are independent random variables having uniform distribution on the interval [0, 2\( \pi \)]. The derived algorithms for Methods 1 and 2 can be easily extended to any space dimension \( M > 2 \).

Given that the random field is homogeneous and isotropic in space, the above-derived algorithms for its simulation significantly reduce the required computational effort, in terms of both computational time and storage requirements. For the \( M = 2 \) case, and, assuming for simplicity, that \( N = N_1 = N_2 \) in either (10) or (11), the simulation of the field as homogeneous only would require the generation of sample functions at a 2D space grid and a computational effort proportional to \( N \log N \). On the other hand, the computational effort for the simulation of a time history at one location for the \( M = 2 \) case, if the isotropy of the field is taken into consideration, and assuming, again, that \( N = N_1 \), is proportional to \( N^2 \). The simulation of time histories at more than one location in space requires minimal additional computational effort, namely, only the repetition of Step 3, for either Method 1 or 2. This is an additional significant advantage of the derived algorithms: They allow the generation of time histories at any arbitrary location \((r, \varphi)\) in the space domain. Such an advantage is of particular importance, e.g., in the simulation of spatially variable seismic ground motions for the seismic response of bridges. For this case, simulated ground motions are required only at the supports of the structure and not over an entire 2D grid of locations, as required when the spectral representation approach for homogeneous fields only is utilized. Indeed, this consideration has been emphasized in the approach for the
simulation of spatially variable seismic ground motions derives from Ramadan and Novak (1993b,c). It is also noted that the simulation of the random field for any space dimension \( M \) as homogeneous only, with the assumption again that \( N = N_1 \equiv \cdots = N_M \) would require a computational effort proportional to \((N \log N)^{M/2+1}\). It can be easily recognized from (14) that for the isotropic case for any value of \( M \), the computational effort is still proportional to \( N^2 \).

**GENERATION OF SPATIALLY VARIABLE SEISMIC GROUND MOTIONS**

In this section, a numerical example illustrating the methodology presented herein is shown. The example deals with the simulation of spatially variable seismic ground motions.

The spatial variability of seismic ground motion plays a significant role in the seismic response of lifelines, such as bridges and pipelines, as it may induce significant additional loads in the structures than the ones induced if the motions at their supports were identical (Zerva 1994). Spatially variable seismic ground motions have been simulated by means of a wide variety of techniques. Examples include, but are not limited to: covariance matrix decomposition (Hao et al. 1989); spectral representation method (Zerva 1992,1994); envelope functions containing random phase variability (Abrahamson 1983); ARMA approximation (Spanos and Zeldin 1996); hybrid DFT and digital filtering approach (Li and Kareem 1997); and conditional simulations (Vanmarcke et al. 1993; Jin et al. 1997).

The description of the spatial variability of seismic ground motions is, generally, based on the statistical analysis of data recorded at dense instrument arrays. It is described by the product of two terms, a real valued one, the coherency that represents the loss of similarity in the seismic motions at various locations on the ground surface, and a complex term that describes the apparent propagation of the motions. An abundance of coherency models have appeared in the literature [see Zerva (1998) for a recent review]. Coherency models assume stationarity in time, and homogeneity and isotropy in space for the random field. The weak anisotropy in seismic recorded data observed by Loh and Lin (1990) and Hao et al. (1989) can be easily modified with a simple transformation (Ramadan and Novak 1993a), so that the isotropy in the random field is preserved. Hence, the consideration of isotropy in the spatially variable random field is valid for simulation purposes.

Spatially variable seismic ground motions at various locations on the ground surface are generated herein with the derived algorithms for homogeneous and partially isotropic random fields. It is noted that only the loss of coherency in the spatial variability of the motions is taken into account in the simulation process. Because the apparent propagation of the motions on the ground surface is direction dependent, it is not incorporated in the cross-spectral density used for the simulations. However, once the motions are simulated, the effect of their propagation can be easily imposed on the time histories by means of an appropriate time delay. Furthermore, it is noted that the simulated seismic ground motions are stationary.

The nonstationary characteristics of the actual seismic ground motions can be incorporated in such simulations generated from stationary random fields by multiplying the stationary signals with slowly varying, deterministic time functions (Ramadan and Novak 1993c).

The ground acceleration field is denoted by \( f(r, \varphi, t) \), where \( r \) and \( \varphi \) are polar coordinates, and \( t \) denotes time. The one-sided cross-spectral density for the homogeneous and partially isotropic ground acceleration field between two locations on the ground surface at a distance \( \rho = |\Delta \mathbf{x}| \), with \( \Delta \mathbf{x} \) used in the derivation of the algorithm) apart from one another is given by

\[
S_s(\rho, \omega) = S_s(\omega)\gamma(\rho, \omega) \tag{23}
\]

where \( S_s(\omega) = \text{Clough-Penzien spectrum given by} \]

\[
1 + 4\zeta_0^2 \left( \frac{\omega}{\omega_0} \right)^2 \left[ 1 - \left( \frac{\omega}{\omega_0} \right)^2 \right] + 4\zeta_0^2 \left( \frac{\omega}{\omega_0} \right)^2 \right]
\]

in which \( S_0 = \text{spectral density of the white noise bedrock excitation; and } \omega_0, \zeta_0, \omega_c, \text{ and } \zeta_c = \zeta_c = 0.6 \text{ (firm soil conditions), and it is assumed that } S_0 = 1 \text{ cm}^2/s. \gamma(\rho, \omega) \text{ in (23) indicates the coherency of the motions. To determine the one-sided isotropic spectral density } S(\kappa, \omega), \text{ we proceed as follows: On one hand, according to the definition of the cross-spectral density, the cross-correlation of the motions between two locations should be}

\[
R_s(\rho, \Delta t) = \int_0^\infty \cos(\omega_0 \Delta t) S_s(\rho, \omega) \, d\omega = \int_0^\infty \cos(\omega_0 \Delta t) S_s(\omega) \gamma(\rho, \omega) \, d\omega
\]

On the other hand, according to (13), the same cross correlation should be equal to

\[
R_s(\rho, \Delta t) = 2\pi \int_0^\infty \int_0^\infty J_0(k\rho) \cos(\omega_0 \Delta t) \kappa S_s(\kappa, \omega) \, d\kappa \, d\omega
\]

where \( J_0 = \text{Bessel function of zero order. Taking the Fourier transform of the last two equations, we obtain}

\[
S_s(\omega) \gamma(\rho, \omega) = 2\pi \int_0^\infty J_0(k\rho) \kappa S_s(\kappa, \omega) \, d\kappa \tag{25}
\]

Using the inverse Fourier-Bessel or Hankel transformation of the zero order (Vilenkin 1968) in (25), we obtain

\[
S_s(\kappa, \omega) = \frac{1}{2\pi} S_s(\omega) \int_0^\infty \gamma(\rho, \omega) J_0(k\rho) \rho \, d\rho \tag{26}
\]

It is noted that the right-hand side of the above equation should be nonnegative. Consequently, a mathematical constraint on the coherency function \( \gamma(\rho, \omega) \) of a homogeneous and partially isotropic random field in \( \mathbb{R}^3 \) is that its Hankel transformation of zero order with respect to \( \rho \) is a nonnegative function.

The coherency model used herein is that developed by Luco and Wong (1986), based on shear-wave propagation through random media

\[
\gamma(\rho, \omega) = \exp(-\alpha^2 \rho^2) \omega^2 \tag{27}
\]

The exponential decay of this model is controlled by the incoherence parameter \( \alpha \). The higher the value of \( \alpha \), the stronger the degree of exponential decay. Luco and Wong (1986) suggested values for \( \alpha \) between 2 and 3 \( \times 10^{-2} \text{ s/m} \), so that the exponential decay of their model fits the exponential decay of the coherency of actual recorded data. Herein the value of \( \alpha = 2.5 \times 10^{-2} \text{ s/m} \) will be used.

Based on (26), it can be shown that the Luco and Wong model is admissible as the coherency of an isotropic field, and
FIG. 1. Variance $E[f^2(r, 0, 0)]$ as Function of Distance $r$ and Number of Bessel Terms $L$: (a) Acceleration; (b) Displacement

(a) acceleration
(b) displacement

FIG. 2. Target and Estimated Auto- and Cross-Spectral Densities for Ground Acceleration Histories at Two Different Locations, $L = 20$: (a) Autospectral Density; (b) Cross-spectral Density

(a) auto-spectral density
(b) cross-spectral density

FIG. 3. Target and Estimated Auto- and Cross-Spectral Densities for Ground Acceleration Histories at Two Different Locations, $L = 5$: (a) Autospectral Density; (b) Cross-spectral Density

(a) auto-spectral density
(b) cross-spectral density
that the spectral density used in the simulation of the homogeneous and partially isotropic random field is

\[ S_{\omega}(\omega) = \frac{1}{4\pi \omega^2} S_{\omega}(\omega) \exp \left( -\frac{\kappa^2}{4\alpha^2 \omega} \right) \] (28)

The simulations in this example were based on Method 2, with parameters: \( \omega_u = 120 \text{ rad/s} \), \( \kappa_u = 2 \times 10^{-2} \pi \text{ rad/m} \), \( N = 1,024 \) and \( N_1 = 512 \). An additional consideration in the simulation of homogeneous and partially isotropic random fields is the selection of an appropriate value for the number of Bessel terms \( L \). As indicated in the derivation of the approach, for finite values of the parameter \( L \), the random field \( f(r, \varphi, t) \) is not homogeneous, but asymptotically homogeneous, due to Neumann’s addition theorem. To illustrate the effect of the selection of \( L \) on the generated motions, we investigate the variance of the random field. For \( \varphi = t = 0 \), the variance of the field (20) becomes

The behavior of the variance for different values of \( L \) and for \( r \) ranging from 0 to 1,000 m is demonstrated in Fig. 1 both for accelerations [Fig. 1(a)] and for displacements [Fig. 1(b)]. It can be seen from the figure that the choice of \( L \) depends on the radius of the area within which we want to simulate the random field. For example, for modeling accelerations (displacements) and for an area of radius of 100 m, \( L = 5 \) (\( L = 1 \)) is sufficient; whereas, for an area of radius 800 m, one should choose \( L = 20 \) (\( L \geq 3 \)).

Fig. 2(a) shows the target and estimated autospectral densities of ground acceleration at one location, \( r = 800 \) m. Fig. 2(b) shows the target and estimated cross-spectral densities between the motions at two different locations 800 m apart from each other, namely, at \( r = 0 \) and 800 m. The estimated
spectra in both cases were obtained from the ensemble averages of 100 realizations, for which $L = 20$ was used. It can be seen from the figures that the estimated spectra are in good agreement with the target ones.

Fig. 3 plots the same results as Fig. 2, but in this case only five, $L = 5$, Bessel terms are used in the simulations. It can be seen from the figure that the estimated spectra deviate significantly from the target spectra, particularly for the autospectral density. This is because, as can be seen from Fig. 1(a), the use of five terms, $L = 5$, is not sufficient to model accelerations at $r = 800$ m.

Fig. 4 presents a comparison of the variability in the seismic ground motions, when their coherency is described by the model of Luco and Wong (1986). Fig. 4(a) presents acceleration time histories at two locations having a separation distance of 100 m, and Fig. 4(b) presents time histories when the separation distance between the two stations increases to 800 m. For the short separation distance, the seismic ground motions are very similar. This is consistent with the characteristics of the random field: For the value of the incoherence parameter used in the simulations ($\alpha = 2.5 \times 10^{-4}$ m/s), and for $\rho = 100$ m, the coherency model of (27), produces high values of correlations in the motions. On the other hand, for long separation distances ($\rho = 800$ m), the values of coherency in (27) decrease rapidly. Thus, the simulated time histories in Fig. 4(b) exhibit a significant loss of coherency.

Fig. 5 illustrates the loss of coherency in the motions over an area ranging between $-500$ and 500 m in both the $x$- and $y$-direction at a fixed moment of time. The gradual change in the amplitudes of the motions at neighboring locations and their significant variability at long separation distances illustrates the consistency of the simulated motions with the coherency model used in the description of the random field.

**SUMMARY AND CONCLUSIONS**

A computationally efficient approach for the simulation of homogeneous and partially isotropic random fields by means of FFT has been developed in this work. The approach is based on the spectral representation method. It is shown that by incorporating the partial isotropy of the field in the simulation algorithm, the computational effort—in terms of computation and storage requirements—for the simulations is significantly reduced, as compared with the case when only the homogeneity in the field is taken into account. The characteristics of the generated sample functions satisfactorily approximate the target ones. An additional, significant advantage of the developed methodology is that it can evaluate, with minimal additional computational effort, simulations at any arbitrary location; considering the field as homogeneous only would require the evaluation of simulations at a predefined grid of locations, irrespective of whether the simulations at all of these locations were necessary or not.

The developed methodology can be applied for the simulation of any homogeneous and partially isotropic random field. As an example application of the approach, spatially variable seismic ground motions experiencing loss of coherency were simulated. The characteristics of the simulations were consistent with those of the random field, thus, further validating the approach.

**ACKNOWLEDGMENTS**

L. S. Katzfogelias acknowledges the support of the Hong Kong Research Grant Council under Grant HKUST 6041/97E. A. Zerva acknowledges the support of the U.S. National Science Foundation under Grant CMS-9725567.

**APPENDIX. REFERENCES**