1. INTRODUCTION
Transfer functions are widely used in optical imaging system analysis. Imaging systems under study need to be linear and space invariant in order for the transfer function description to be valid. For coherent imaging systems, the object–image relationship is linear in field amplitude, and space invariance can be readily attained by illuminating the object with a converging wave-front focused at the center of the objective lens pupil [1]. Partially coherent imaging systems, on the other hand, are linear in mutual intensity and are not space invariant under critical and Köhler illuminations. The existence of quadratic phase factors in the partially coherent amplitude spread function and the illumination mutual intensity prevents the exact application of transfer functions for partially coherent imaging systems.

Partially coherent imaging systems are only approximately space invariant when the quadratic phase factors in the amplitude spread function and the illumination mutual intensity can be approximated by unity over the range of separation of interest [2]. To satisfy this condition, the coherence area of the illumination needs to be small with respect to the object size [3]. The effect of the quadratic phase terms in this case can be ignored and an approximate transfer function description can be obtained for the partially coherent imaging system. Alternatively, an extra phase-correcting lens of proper focal length can be placed in front of the object to completely eliminate the quadratic phase terms and render the partially coherent imaging system exactly space invariant. However, for short wavelength imaging systems these techniques, i.e., adding extra phase-correcting lens or setting up low coherence approximation, are often difficult in practice. Specifically, the option of installing an extra x-ray lens is impractical due to their highly absorptive nature, and the low coherence approximation is often unrealistic, especially for x-ray interferometric phase imaging systems where a high degree of coherence is generally required [4–9].

Here we propose a new illumination scheme that achieves space invariance and simplifies system analysis by removing the quadratic phase terms in both the amplitude spread function and the illumination mutual intensity. Section 2 proved rigorously that the space invariance condition proposed by Tichenor and Goodman for coherent imaging systems [1] is also valid under partially coherent illumination. However, while this condition indeed renders the amplitude spread function free of quadratic phase factors and the system space invariant, quadratic phase terms still exist in the illumination mutual intensity. As a result, no analytical form exists for the Fourier transform of illumination mutual intensity and one cannot obtain a simple analytical expression for the image intensity distribution. Note that although Köhler illumination does not have quadratic phase factors in its illumination mutual intensity, the square phase terms in its amplitude spread function still render the overall imaging system space variant. Section 3 introduced the additional requirement on the space invariant partially coherent imaging system, i.e., the distance between the condenser and the object needs to be set at the condenser focal length. This condition, together with the space invariance condition described in Section 2, defines the proposed illumination apparatus. Quadratic phase factors are completely eliminated in this case for both the illumination and the amplitude spread function. This new illu-
minimization scheme therefore achieves space invariance and at the same time allows the image intensity distribution to be analytically determined. This illumination method is applicable in general at all wavelengths. No phase-correcting lenses are needed and no restriction on illumination mutual coherence is required except quasimonochromatic and paraxial approximations. The mathematical forms for the amplitude spread function, illumination mutual intensity, and overall image intensity are presented analytically. A special case in which an incoherent source is used for illumination is also discussed.

2. PARTIALLY COHERENT IMAGING SYSTEMS

Figure 1 depicts a typical partially coherent imaging system where the object is illuminated via a condenser and imaged by an objective lens onto the image plane. Coordinate systems for the source, condenser, object, objective lens, and image planes, together with their relative distances along the optical axis, are defined and used throughout this article. Specifically, the relative axial distances between the source, condenser, object, objective lens, and the image planes are designated as $z_1$, $z_2$, $z_o$, and $z_i$. Transverse coordinate systems for these elements are defined as $(\alpha, \beta)$, $(\xi, \eta)$, $(x, y)$, and $(u, v)$, respectively. The propagation of mutual intensity is examined in cascade from the source to the image plane.

A. Illumination

The illumination section is composed of a partially coherent source and a condenser. Object illumination is derived in cascade by propagating the mutual intensity from the source to the object plane. Given a partially coherent source, under quasimonochromatic and paraxial approximations mutual intensity incident into the condenser is given by $[3,10]$

$$J_c(x_1, y_1; x_2, y_2) = \frac{1}{(\lambda z_1)^2} \exp \left\{ -\frac{j \pi}{\lambda z_1} [\bar{x}_2^2 + \bar{y}_2^2 - \bar{x}_1^2 - \bar{y}_1^2] \right\}$$

$$\times \int \int \int J_1(\alpha_1, \beta_1; \alpha_2, \beta_2) \times \exp \left\{ -\frac{j \pi}{\lambda z_1} [\alpha_2^2 + \beta_2^2 - \alpha_1^2 - \beta_1^2] \right\} \times \exp \left\{ \frac{2\pi}{\lambda z_1} [\bar{x}_2 \alpha_2 + \bar{y}_2 \beta_2 - \bar{x}_1 \alpha_1 - \bar{y}_1 \beta_1] \right\} \, d\alpha_1 d\beta_1 d\alpha_2 d\beta_2,$$

(1)

where $\lambda$ is the wavelength, $J_c(x_1, y_1; x_2, y_2)$ is the mutual intensity in front of the condenser, and $J_1(\alpha_1, \beta_1; \alpha_2, \beta_2)$ is that of the source. Using a large and aberration-free thin lens as the condenser, the mutual intensity leaving the back of the condenser $J'_c$ is given by

$$J'_c(x_1, y_1; x_2, y_2) = J_c(x_1, y_1; x_2, y_2) \times \exp \left\{ \frac{\pi}{\lambda f_c} [\bar{x}_2^2 + \bar{y}_2^2 - \bar{x}_1^2 - \bar{y}_1^2] \right\},$$

(2)

where $f_c$ is the condenser focal length. Note that the condenser lens aperture is large enough such that the extent of $J_c$ is not constricted as it passes the condenser. The propagation of mutual coherence from the condenser to the object plane is given by
\[ J_0(\xi_1, \eta_1; \xi_2, \eta_2) = \frac{1}{(\lambda z_2)^2} \exp \left\{ -j \frac{\pi}{\lambda z_2} \left[ \xi_2^2 + \eta_2^2 - \xi_1^2 - \eta_1^2 \right] \right\} \times \int \int \int \int J'_1(\tilde{x}_1, \tilde{y}_1; x_2, y_2) \times \exp \left\{ -j \frac{\pi}{\lambda z_2} \left[ \xi_2^2 + \eta_2^2 - \xi_1^2 - \eta_1^2 \right] \right\} \] and the mutual intensity incident into the object can therefore be obtained by combining Eqs. (1)–(3) as

\[ J_0(\xi_1, \eta_1; \xi_2, \eta_2) = \frac{1}{(\lambda z_2)^2} \exp \left\{ -j \frac{\pi}{\lambda z_2} \left[ \xi_2^2 + \eta_2^2 - \xi_1^2 - \eta_1^2 \right] \right\} \times \int \int \int \int d\alpha_1 d\beta_1 d\alpha_2 d\beta_2 J_1(\alpha_1, \beta_1; \alpha_2, \beta_2) \times \exp \left\{ -j \frac{\pi}{\lambda} \left[ \frac{1}{z_1} + \frac{1}{z_2} - \frac{1}{f_c} \right] \right\} \exp \left\{ -j \frac{\pi}{\lambda \frac{1}{z_1} + \frac{1}{z_2} - \frac{1}{f_c}} \right\} \left( \frac{1}{z_2^2} \right) \exp \left\{ -j \frac{\pi}{\lambda} \left[ \frac{1}{z_1} + \frac{1}{z_2} - \frac{1}{f_c} \right] \right\} \]}

Using the identity

\[ \int \exp \left\{ \pm j \frac{\pi}{\lambda} x^2 \right\} \exp(xj2\pi v^2) dx = \sqrt{j\lambda f} \exp(xj\pi v^2), \]

the mutual intensity incident into the sample in Eq. (4) can be rewritten as

\[ J_0(\xi_1, \eta_1; \xi_2, \eta_2) = \frac{1}{\lambda z_2^2} \left( \frac{1}{z_1} + \frac{1}{z_2} - \frac{1}{f_c} \right) \exp \left\{ -j \frac{\pi}{\lambda z_2} \left[ \xi_2^2 + \eta_2^2 - \xi_1^2 - \eta_1^2 \right] \right\} \exp \left\{ -j \frac{\pi}{\lambda} \left[ \frac{1}{z_1} + \frac{1}{z_2} - \frac{1}{f_c} \right] \right\} \exp \left\{ -j \frac{\pi}{\lambda} \left[ \frac{1}{z_1} + \frac{1}{z_2} - \frac{1}{f_c} \right] \right\} \left( \frac{1}{z_2^2} \right) \exp \left\{ -j \frac{\pi}{\lambda} \left[ \frac{1}{z_1} + \frac{1}{z_2} - \frac{1}{f_c} \right] \right\} \]}

Note that \( 1/z_1 + 1/z_2 - 1/f_c \neq 0 \) in this case, and when \( z_1 = z_2 = f_c \) the above equation reduces to the well-known formula for the propagation of mutual intensity from the front to the rear focal planes of a thin lens [3].

B. Image Formation

A thin objective lens is used to form a conjugate image of the object, which is trans-illuminated by \( J_0 \) given in Eq. (5). Note that here the object and image lengths, \( z_0 \) and \( z_i \) as defined in Fig. 1, satisfy the Gaussian lens formula, \( 1/z_0 + 1/z_i - 1/f_m = 0 \), where \( f_m \) is the focal length of the objective lens. The coherence relationship between an object-image conjugate pair of a thin objective lens can therefore be described as [3]

\[ J_i(u_1, v_1; u_2, v_2) = \int \int \int J_0(\xi_1, \eta_1; \xi_2, \eta_2) t_i(\xi_1, \eta_1) \times t_i(\xi_2, \eta_2) K(u_1, v_1; \xi_1, \eta_1) K^*(u_2, v_2; \xi_2, \eta_2) \times d\xi_1 d\eta_1 d\xi_2 d\eta_2, \]

where \( J_i \) is the mutual intensity at the image plane, \( t_i \) is the amplitude transmittance of a trans-illuminated object, and the amplitude spread function \( K \) is defined as
by the objective lens in the image formation section of the microscope, the distance from the object to the objective lens \( z_o \) can be chosen such that the quadratic phase terms in the amplitude spread function are matched to those in the illumination. The overall imaging system is then space invariant. Putting Eqs. (5) and (7) into Eq. (6), the quadratic phase terms in \( (\xi_1, \eta_1; \xi_2, \eta_2) \) are indeed canceled if the following condition is satisfied:

\[
\frac{1}{z_1} + \frac{1}{z_2 + z_o} = \frac{1}{z_1}.
\]

Note that given Eq. (8) a conjugate image of the source is formed at a distance of \((z_2 + z_o)\) behind the condenser. This space invariance condition for a partially coherent imaging system is essentially identical to that for a coherent imaging system [1], i.e., both coherent and partially coherent imaging systems require a converging wave-front focused on the objective lens aperture in order to be space invariant.

The image mutual intensity \( J_i \) in this case can be expressed as

\[
J_i(u_1,v_1;u_2,v_2) = \exp \left\{ \frac{\pi}{\lambda z_1} (u_1^2 + v_1^2) \right\} \exp \left\{ -\frac{\pi}{\lambda z_2} (u_2^2 + v_2^2) \right\}
\]

\[
\times \int \int P(x,y) \exp \left\{ -\frac{2\pi}{\lambda z_1} \left( u + \frac{z_1}{z_o} \right) x \right\}
\]

\[
\times \left\{ e^{i \left( v + \frac{z_1}{z_o} \right) y} \right\} dx dy,
\]

respectively. It is important to note that Eqs. (10) and (11) need to be applied concurrently in order to ensure the cancellation of their respective quadratic phase terms. Only under this circumstance is the space invariant amplitude spread function \( K_{\text{eff}} \) applicable for the overall imaging system. Typical partially coherent imaging systems
tors in the amplitude spread function.

The quadratic phase terms in front of the integral in Eq. (9), i.e., \( \exp(j \pi / \lambda z)(u_1^2 + v_1^2) \) and \( \exp(-j \pi / \lambda z)(u_2^2 + v_2^2) \), can be ignored without loss of intensity information since these phase terms disappear in image intensity distribution, i.e., when setting \((u_1, v_1) = (u_2, v_2) = (u, v)\). All that remained between Eq. (9) and a convolution integral is a simple scaling factor in the coordinate system, i.e., the magnification. To make the equations formally space invariant, one defines the magnification as \( M \triangleq z_z/z_o \), then \( \xi \triangleq -(z_z/z_o) \xi = -M \xi \) and \( \eta \triangleq -M \eta \). The amplitude spread function \( K_{eff}(u, v; \xi, \eta) \) in Eq. (11) can now be written as

\[
K_{eff}(u, v; \xi, \eta) \triangleq K(u - \xi', v - \eta') \frac{1}{\lambda^2 z_z z_o} \int \int \int P(x, y) \times \exp\left\{ -j \frac{2 \pi}{\lambda z_i} [(u - \xi')x + (v - \eta')y] \right\} dx dy. \tag{12}
\]

The integral for the image mutual intensity \( J \) is now a 4D convolution,

\[
J_i(u_1, v_1; u_2, v_2) = \int \int \int \int \frac{d \xi_1 d \eta_1 d \xi_2 d \eta_2}{M^4} \times J_0(\xi_1, \eta_1; \xi_2, \eta_2)K(u_1 - \xi_1, v_1 - \eta_1)K'(u_2 - \xi_2, v_2 - \eta_2), \tag{13}
\]

where

\[
J_0(\xi_1, \eta_1; \xi_2, \eta_2) \triangleq J_{s,eff}(\xi_1, \eta_1; \xi_2, \eta_2)T_0(\xi_1, \eta_1)T_0(\xi_2, \eta_2) \text{ is expressed in the }(\xi', \eta') \text{ coordinate. Note that } T_0(\xi', \eta') \text{ is the geometric image of the sample as viewed at the image plane.}
\]

The 4D Fourier transform of the convolution integral in Eq. (13) is simply

\[
J_0(\xi_1, \eta_1; \xi_2, \eta_2) = J'_{s,eff}(\xi_1, \eta_1; \xi_2, \eta_2)K_0(\xi_1, \eta_1)K'_0(\xi_2, \eta_2), \tag{14}
\]

where

\[
K(p, q) = MP(\lambda z_o M_p, \lambda z_o M_q) \tag{15}
\]

is the transfer function given by the Fourier transform of Eq. (12), and \( J_i \) and \( J_0 \) are the 4D Fourier spectra of \( J_i \) and \( J_{s,eff} \), respectively. Note that even though the overall imaging system is indeed space invariant, \( J_{s,eff} \) in Eq. (10) is not a Fourier transform of \( J_i \).

3. NEW SPACE INVARIANT ILLUMINATION METHOD

We now proceed to defining the second condition for the proposed illumination apparatus. Examining Eq. (10) again one finds that even with the space invariant condition in Eq. (8), the quadratic phase terms within the integrand still prevent the object illumination \( J_{s,eff} \) from being a 4D Fourier transform of the source mutual intensity \( J_i \). As a result, the use of transfer function method in Eq. (14) is limited in that the image intensity distribution cannot be analytically determined due to the lack of analytical expressions for the Fourier transforms of \( J_{s,eff} \) and \( J_i \). Nevertheless, when the distance between the condenser and the object is fixed at the focal length of the condenser, i.e.,

\[
z_2 = f_c, \tag{16}
\]

the square phase terms in \((\alpha_1, \beta_1; \alpha_2, \beta_2)\) are canceled and the 4D integral now involves only the cross-terms between \((\alpha_1, \beta_1)\) and \((\xi, \eta)\), i.e.,

\[
\exp[(2 \pi / \lambda f_c)(\alpha_2 \xi_2 + \beta_2 \eta_2 - \alpha_1 \xi_1 - \beta_1 \eta_1)]. \tag{17}
\]

Equation (8), or equivalently Eq. (17), together with Eq. (16), defines the new illumination apparatus proposed here for space invariant partially coherent imaging systems. Its illumination mutual intensity is obtained by putting Eqs. (16) and (17) into Eq. (10), i.e.,

\[
J_{s,eff}(\xi_1, \eta_1; \xi_2, \eta_2) = \frac{1}{(\lambda f_c)^2} \int \int \int J_s(\alpha_1, \beta_1; \alpha_2, \beta_2) \times \exp\left\{ j \frac{2 \pi}{\lambda f_c} [\alpha_2 \xi_2 + \beta_2 \eta_2 - \alpha_1 \xi_1 - \beta_1 \eta_1] \right\} d\alpha_1 d\beta_1 d\alpha_2 d\beta_2. \tag{18}
\]

The object illumination is now the 4D Fourier transform of the source mutual intensity. In the next paragraph, we show that with this new illumination method the transfer function equation in Eq. (14) can be used to obtain the image intensity directly.

Fourier transform of the illumination mutual intensity in Eq. (18) is given by

\[
J_0(p', q', r', s') = (M^2 \lambda f_c)^2 J_0(-\lambda f_c M_p', -\lambda f_c M_q', \lambda f_c M_r', \lambda f_c M_s'). \tag{19}
\]

Note that following the notations in Eq. (13), mutual intensities incident into and leaving the object, as well as the object’s amplitude transmittance functions, must be expressed in the \((\xi', \eta')\) coordinate in order to be formally space invariant. The 4D Fourier transform of the mutual intensity leaving the object is, therefore,

\[
J_0'(v_1, v_2, v_3, v_4) = \int \int \int J_0(p', q', r', s') T_0(\xi_1 - p', \xi_2 - q') - q') T_0(\eta_1 - s', \eta_2 - v' d\xi' d\eta' d\xi d\eta d\xi' d\eta', \tag{20}
\]

where \( T_0 \) is the Fourier transform of the object’s amplitude transmittance as seen in the image plane. The image intensity can then be obtained by performing an inverse Fourier transform of the transfer function equation in Eq. (14) and setting \((u_1, v_1) = (u_2, v_2) = (u, v)\) (see Appendix A), i.e.,
shows that object illumination provided by the proposed illumination apparatus with an incoherent source therefore constitutes an “inverted Köhler” illumination, which satisfies the space invariance condition (8) with $z_2 = f_c$, while the Köhler illumination is space variant with $z_1 = f_c$.

The image intensity in this case is obtained by putting Eqs. (22) and (19) into Eq. (21) and integrating over the $\delta$-function (see Appendix B), i.e.,

$$I_i(u,v) = \int_{-\infty}^{\infty} dp dq \int_{-\infty}^{\infty} dw_2 T_o(w_1, w_2) K(w_1 + p, w_2 - q)$$

where $J_o$ is given by the source mutual intensity in Eq. (19) and $K$ is given by the pupil function of the objective lens in Eq. (15). We have therefore shown that the proposed illumination, as defined by Eqs. (8) and (16), renders the image formation process space invariant and enables exact transfer function description of partially coherent imaging systems.

A. Special Case: Illumination Using an Incoherent Source

For partially coherent imaging systems, an incoherent source is most commonly employed for illumination and in this case the expression for the image intensity [Eq. (21)] can be further simplified. The mutual intensity of an incoherent source is given by [3]

$$J_o(\alpha_1, \beta_1; \alpha_2, \beta_2) = \kappa \delta(\alpha_1 - \alpha_2, \beta_1 - \beta_2), \quad (22)$$

where $\kappa = \lambda^2 / \pi$ and $I_o$ is the source intensity distribution. The mutual intensity incident into the object is obtained from Eq. (18) as

$$J_o(\Delta \xi, \Delta \eta) = \frac{\kappa}{(4\pi)^2} \int_{-\infty}^{\infty} I_o(\alpha, \beta) \exp\left\{ \frac{2\pi i}{\lambda f_c} (\alpha \Delta \xi + \beta \Delta \eta) \right\} d\alpha d\beta, \quad (23)$$

where $(\Delta \xi, \Delta \eta) = (\xi_2 - \xi_1, \eta_2 - \eta_1)$. Note that due to the existence of a $\delta$-function in the source mutual intensity, the illumination in Eq. (23) becomes a two-dimensional instead of the original 4D integral in Eq. (18). Equation (23) shows that object illumination provided by the proposed illumination apparatus is, in format, identical to that of Köhler in this special case of an incoherent source. Indeed, in a typical Köhler illumination setup [2] an auxiliary lens forms a geometric image of the incoherent source at the condenser aperture stop. This geometric image then serves as a secondary source for the condenser. The aperture stop is placed at the front focal plane of the condenser, i.e., $z_1 = f_c$, and its size is controlled by an iris diaphragm. When the iris diaphragm at the condenser aperture stop is opened large enough, and assuming an aberration-free condenser pupil, the range of available angular illumination onto the object is determined by both the dimension of the secondary source and the focal length of the condenser [3]. The proposed illumination apparatus with an incoherent source therefore constitutes an “inverted Köhler” illumination, which satisfies the space invariance condition (8) with $z_2 = f_c$, while the Köhler illumination is space variant with $z_1 = f_c$.

4. CONCLUSION

The proposed illumination apparatus for partially coherent imaging systems achieves space invariance and permits exact transfer function analysis without the need for additional phase-correcting lenses or approximation on illumination mutual coherence. The resultant optical transfer function of the imaging system therefore is not prone to the additional errors due to the approximations on quadratic phase terms in the amplitude transfer function and illumination coherence. Better agreement between experimental and theoretical results can potentially be attained for high resolution x-ray microscopy [6]. With the relentless push for ever higher spatial resolution in x-ray microscopy [6,11,12], we expect that the space invariant illumination apparatus proposed here will play an important role as precise characterization of partially coherent imaging systems becomes inevitable for both phase and absorption contrast mechanisms [13–17].

APPENDIX A: IMAGE INTENSITY DISTRIBUTION USING THE NEW ILLUMINATION APPARATUS

This appendix details the derivation of the image intensity distribution given in Eq. (21). The effective illumination mutual intensity $J_{o,eff}$ is given by Eq. (18) as
respectively. Equation (A5) can be further simplified by substituting

\[
\begin{align*}
J_{o,eff}(\xi_1; \eta_1; \xi_2, \eta_2) &= \frac{1}{(\lambda f_c)^2} \int \int \int J_i(\alpha_1, \beta_1; \alpha_2, \beta_2) \\
&\times \exp\left\{ \frac{2\pi}{\lambda f_c} [\alpha_2 \xi_2 + \beta_2 \eta_2 - \alpha_1 \xi_1 - \beta_1 \eta_1] \right\}.
\end{align*}
\]

(A2)

The Fourier transform of the illumination mutual intensity in Eq. (A3) is, therefore,

\[
J_o(p', q', r', s') = \int \int \int d\xi_1 d\eta_1 d\xi_2 d\eta_2 J_i(\xi_1; \eta_1; \xi_2, \eta_2) \exp\{j2\pi(\xi_1 p' + \eta_1 q' + \xi_2 r' + \eta_2 s')\}
\]

\[
= \int \int \int d\alpha_1 d\beta_1 d\alpha_2 d\beta_2 J_i(\alpha_1, \beta_1; \alpha_2, \beta_2) \int \int \int d\xi_1 d\eta_1 d\xi_2 d\eta_2 \exp\left\{ j2\pi \left( \xi_1 \left( p' + \frac{\alpha_1}{\lambda f_c} \right) \right) \\
+ \eta_1 \left( q' + \frac{\beta_1}{\lambda f_c} \right) + \xi_2 \left( r' - \frac{\alpha_2}{\lambda f_c} \right) + \eta_2 \left( s' - \frac{\beta_2}{\lambda f_c} \right) \right\}
\]

\[
= (M^2 \lambda f_c)^2 J_i(-\lambda f_c p', -\lambda f_c q', -\lambda f_c r', -\lambda f_c s').
\]

(A3)

The mutual intensity leaving the object is given by

\[
J_o(\xi_1; \eta_1; \xi_2, \eta_2) = J_o(\xi_1; \eta_1; \xi_2, \eta_2) t_o(\xi_1, \eta_1) t^*_o(\xi_2, \eta_2),
\]

(A4)

and its Fourier transform is, thus,

\[
J_o(v_1, v_2, v_3, v_4) = \int \int \int d\xi_1 d\eta_1 d\xi_2 d\eta_2 J_i(\xi_1; \eta_1; \xi_2, \eta_2) t_o(\xi_1, \eta_1) t^*_o(\xi_2, \eta_2) \exp\{j2\pi(\xi_1 v_1 + \eta_1 v_2 + \xi_2 v_3 + \eta_2 v_4)\},
\]

(A5)

with \(J_o\), \(t_o\), and \(t^*_o\) expressed as

\[
J_o(\xi_1; \eta_1; \xi_2, \eta_2) = \int \int \int dp' dq' ds' J_o(p', q', r', s') \exp\{-j2\pi(p' \xi_1 + q' \eta_1 + r' \xi_2 + s' \eta_2)\},
\]

(A6a)

\[
t_o(\xi_1, \eta_1) = \int \int d\alpha' d\beta' T_o(\alpha', \beta') \exp\{-j2\pi(\alpha' \xi_1 + \beta' \eta_1)\},
\]

(A6b)

\[
t^*_o(\xi_2, \eta_2) = \int \int dx' dy' T^*_o(x', y') \exp\{-j2\pi(x' \xi_2 + y' \eta_2)\},
\]

(A6c)

respectively. Equation (A5) can be further simplified by substituting \(J_o\), \(t_o\), and \(t^*_o\) in Eqs. (A6), i.e.,
and setting

The image intensity can now be obtained by substituting Eq. (A7) into Eq. (14), performing the inverse Fourier transform, and setting \((u_1, v_1) = (u_2, v_2) = (u, v)\), i.e.,

\[
I_i(u, v) = \int \int \int \int dp' dq' dr' ds' J_o(p', q', r', s') \int \int \int dx' dy' T_o(x', y')
\times \int \int \int \int d\xi_1 d\eta_1 d\xi_2 d\eta_2 \exp{j2\pi[\xi_1^2 (v_1 - p' - \alpha') + \eta_1^2 (v_2 - q' - \beta') + \xi_2^2 (v_3 - r' + x') + \eta_2^2 (v_4 - s' + y')]}
\]

\[
= \int \int \int \int J_o(p', q', r', s') T_o(v_1 - p', v_2 - q') T_o^*(r' - v_3, s' - v_4) dp' dq' dr' ds'.
\]  

(A7)

The image intensity can now be obtained by substituting Eq. (A7) into Eq. (14), performing the inverse Fourier transform, and setting \((u_1, v_1) = (u_2, v_2) = (u, v)\), i.e.,

\[
I_i(u, v) = \int \int \int \int d\nu_1 d\nu_2 d\nu_3 d\nu_4 \exp[-j2\pi (\nu_1 u + \nu_2 v + \nu_3 u + \nu_4 v)] J_o(\nu_1, \nu_2, \nu_3, \nu_4) K(\nu_1, \nu_2) K^*(-\nu_3, -\nu_4)
\]

\[
= \int \int \int \int dp' dq' dr' ds' J_o(p', q', r', s') \int \int \int \int d\nu_1 d\nu_2 d\nu_3 d\nu_4 \exp[-j2\pi u(\nu_1 + \nu_2)
\]

\[
+ v(\nu_2 + \nu_4)] K(\nu_1, \nu_2) T_o(\nu_1 - p', \nu_2 - q') K^*(-\nu_3, -\nu_4) T_o^*(r' - v_3, s' - v_4),
\]  

(A8)

where \(K\) is given in Eq. (15). With a change of integration variables, i.e.,

\[
w_1 = v_1 - p',
\]

\[
w_2 = v_2 - q',
\]

\[
w_3 = v_3,
\]

\[
w_4 = v_4,
\]

the Jacobian determinant is 1 and Eq. (A8) becomes

\[
I_i(u, v) = \int \int \int \int dp' dq' dr' ds' J_o(p', q', r', s') \int \int \int \int dw_1 dw_2 T_o(w_1, w_2) \exp[-j2\pi (u w_1 + v w_2) + p' r' + q' s']
\]

\[
\times \exp[-j2\pi (u w_1 + v w_2)] T_o^*(w_1 - v_1 + p', w_2 - v_2 + q' + s') K^*(w_1 - v_1 + p', w_2 - v_2 + q'),
\]  

(A9)

where \(\kappa = \lambda^2 / \pi\) and \(I_i\) is the source intensity distribution. Substituting Eq. (B1) into Eq. (A3), we obtain

\[
J_o(p', q', r', s') = \kappa (M^2 M_p' - \lambda_f M_q') \delta - \lambda_f M(r' + p'), - \lambda_f M(s' + q').
\]  

(B2)

Putting Eq. (B2) into Eq. (A9), the image intensity in this case is given by

\[
J_s(\alpha_1, \beta_1; \alpha_2, \beta_2) = \kappa J_s(\alpha_1, \beta_1) \delta (\alpha_1 - \alpha_2, \beta_1 - \beta_2),
\]

(B1)
\[ I_i(u, v) = \kappa |M|^2 (\kappa f)^2 \int \int dp' dq' I_i(-\kappa f M p', -\kappa f M q') \int \int dw_1 dw_2 \mathcal{T}_\omega (w_1, w_2) \mathcal{K}(w_1 + p', w_2 + q') \int \int dv_1 dv_2 \]

\[
\times \exp[-j2 \pi (u v u' + v v v')] \mathcal{K}(w_1 - v u' + p', w_2 - v v' + q') \int \int dr' ds' \mathcal{T}_\omega (w_1 - v u' + p' + r', w_2 - v v' + q' + s') \delta(-\kappa f M (r' + p'), -\kappa f M (s' + q')).
\]

(B3)

Integrating over \((r', s')\), the image intensity becomes

\[
I_i(u, v) = \kappa |M|^2 \int \int dp' dq' I_i(\kappa f M p', -\kappa f M q') \int \int dw_1 dw_2 \mathcal{T}_\omega (w_1, w_2) \mathcal{K}(w_1 + p', w_2 + q') \int \int dv_1 dv_2 \exp[-j2 \pi (u v u' + v v v')] \mathcal{T}_\omega (w_1 - v u', w_2 - v v' + q' - v v') \mathcal{K}(w_1 - v u', w_2 + q' - v v').
\]

(B4)

Letting \(p' = -p\) and \(q' = -q\), we have

\[
I_i(u, v) = \int \int dp dq J_o(p, q) \int \int dw_1 dw_2 \mathcal{T}_\omega (w_1, w_2) \mathcal{K}(w_1 - p, w_2 - q) \int \int dv_1 dv_2 \exp[-j2 \pi (u v u' + v v v')] \mathcal{T}_\omega (w_1 - v u', w_2 - v v' + q' - v v') \mathcal{K}(w_1 - v u', w_2 + q' - v v'),
\]

(B5)

where \(J_o\) is given by

\[
J_o(p, q) = \kappa |M|^2 I_i(\kappa f M p, \kappa f M q).
\]

(B6)

Note that these are indeed Eqs. (24) and (25) in the main text.

REFERENCES