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Limit distribution for the maximum degree of a random recursive tree

William Goh, Eric Schmutz *

MCS Department, Drexel University, Philadelphia, PA 19104, USA

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Abstract

If a recursive tree is selected uniformly at random from among all recursive trees on n vertices, then the distribution of the maximum in-degree Δ is given asymptotically by the following theorem: for any fixed integer d,

$$P_n(\Delta \leqslant \lfloor \mu_n \rfloor + d) = \exp(-2^{\{\mu_n\} - d - 1}) + o(1)$$

as $n \to \infty$, where $\mu_n = \log_2 n$. (As usual, $\lfloor \mu_n \rfloor$ denotes the greatest integer less than or equal to μ_n, and $\{\mu_n\} = \mu_n - \lfloor \mu_n \rfloor$.) The proof makes extensive use of asymptotic approximations for the partial sums of the exponential series.

Keywords: Asymptotic enumeration; Degree; Recursive; Szegő; Exponential series; Tree

1. Introduction

A recursive tree on n vertices is one of the $(n-1)!$ directed graphs that can be constructed as follows: label n vertices with the numbers $1, 2, \ldots, n$, and then for $v = 2, 3, 4, \ldots, n$, add a directed edge joining v to some $i < v$. The distinctive property of these trees is that the labels on any path are decreasing. Many authors have studied random recursive trees, e.g., Balinska and Szymanski [1], Dobrow [7], Dondajewski and Szymanski [9], Dondajewski et al. [10], Dobrow and Fill [8], Devroye [4,5], Lu and Devroye [6], Gastwirth and Bhattacharya [2], Mahmoud [12], Mahmoud and Smythe [14,15], Meir and Moon [17–20], Moon [21], Na and Rapoport [22], Pittel [23], and Szymanski [24–26]. Mahmoud and Smythe [16] review the literature on recursive trees.

Let ρ_v be the in-degree of v, and define $\Delta = \max_v \rho_v$. Szymanski was apparently the first to ask how large $\Delta = \Delta(T)$ is for a typical recursive tree T. In [25], he proved that $(1-\varepsilon)\ln n \leqslant \Delta \leqslant \log_2 n$

* Corresponding author.

E-mail addresses: wgoh@mcs.drexel.edu (W. Goh), eschmutz@mcs.drexel.edu (E. Schmutz).

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for all but $o((n-1)!)$ recursive trees on n vertices. Devroye and Jiang Lu [6] strengthened this result using probabilistic methods. For the uniform distribution P_n, they proved that $\Delta/\log_2 n \to 1$ in probability.

Because of the way recursive trees are grown, it is natural to write ρ_v as a sum of indicators: $\rho_v = \sum_{j=v+1}^{n} I_{j,v}$, where $I_{j,v}(T)=1$ iff T has an edge from j to v. Note that, for each v, the indicators $I_{v+1,v}, I_{v+2,v}, \ldots, I_{n,v}$ are independent, and that $P_n(I_{j,v}=1)=1/(j-1)$. It is therefore clear that, for fixed or slowly growing v, ρ_v is asymptotically normal with mean and variance both $\log(n/v)+O(1)$. The random variables ρ_v $(v=1,2,\ldots)$ are not independent, since $I_{j,v}I_{j,w}=0$ for $v \neq w$. Nevertheless, $\mathrm{Cov}(\rho_v,\rho_w)=O(1)$, so it is reasonable to conjecture that the ρ_v's behave like independent normals. The maximum of weakly dependent normal variables frequently has a limiting Gumbel law [13]. However, the ρ_v's are not identically distributed, and the asymptotic distribution of Δ was not obvious to us. The main result in this paper is the following theorem.

Theorem 1. *For any fixed integer d,*

$$P_n(\Delta \leqslant \lfloor \log_2 n \rfloor + d) = \exp(-2^{\{\log_2 n\}-d-1}) + o(1)$$

as $n \to \infty$.

To prove Theorem 1, we need asymptotic approximations for

$$S_k(x) = \sum_{m=0}^{k} \frac{x^m}{m!}. \tag{1}$$

To simplify lengthy formulas, define $Z(k,t)=(1/\sqrt{2\pi k})(t/(1-t))(te^{1-t})^k$. Then Szegő's original estimate [27] is the following theorem.

Theorem 2. *If D is a compact subset of the unit disk, then*

$$e^{-kt} S_k(kt) = 1 - Z(k,t)\left(1 + O\left(\frac{1}{k}\right)\right)$$

uniformly for all $t \in D$.

Several authors have worked to extend Szegő's result, e.g., Temme [28]. We need the following variant that Wimp proved [30].

Theorem 3.

$$S_k(kx) = e^{kx}\left(\delta(x) + \sqrt{\frac{2}{\pi}}\,\frac{\xi x}{x-1}\, Erfc(\sqrt{k}\xi)\left(1 + O\left(\frac{1}{\sqrt{k}}\right)\right)\right)$$

uniformly for $x \geqslant 0$, where $\xi = |x-1-\ln x|^{1/2}$, and where $\delta(x)=1$ for $0 \leqslant x < 1$, and $\delta(x)=0$ for $x \geqslant 1$, and for all t $Erfc(t)=\int_t^{\infty} e^{-s^2/2}\,ds$.

2. Generating functions and the radius of convergence

Let $Y(x)=\sum_{n=1}^{\infty}(y_n x^n)/n!$, where y_n is the number of recursive trees on n vertices. We know $y_n=(n-1)!$, so $Y(x)=\sum_{n\geqslant 1}x^n/n=\log(1/(1-x))$. Similarly, for any positive integer k, let $y_{n,k}$ be the number of recursive trees on n vertices with the property that the maximum in-degree is less than or equal to k, and let $Y_k(x)=\sum_{n=1}^{\infty}(y_{n,k}x^n)/n!$. In order to do asymptotics, we need the fact that Y_k satisfies a simple differential equation.

Lemma 4. $Y_k'=S_k(Y_k)$.

Proof. Use the standard "delete the root" correspondence between trees on $n+1$ vertices and forests on n vertices: the coefficient of x^n in $n!(Y_k(x)^m/m!)$ is the number of recursive trees on $n+1$ vertices in which the root has in-degree m and all other vertices have in-degree less than or equal to k. Hence

$$\sum_{n\geqslant 0}\frac{y_{n+1,k}}{n!}x^n=\sum_{m=0}^{k}\frac{Y_k(x)^m}{m!}. \qquad \square$$

Let r_k denote the radius of convergence of Y_k. Observe that $r_k\geqslant 1$ since $y_{n,k}\leqslant y_n$ for all n and k, and the radius of convergence of Y is 1. Therefore, by Cauchy's integral formula,

$$P_n(\Delta\leqslant k)=\frac{y_{n,k}}{(n-1)!}=\frac{n}{2\pi\mathrm{i}}\oint_{C_k}\frac{Y_k(x)}{x^{n+1}}\,\mathrm{d}x=\frac{1}{r_k^n}J_{n,k}, \tag{2}$$

where $J_{n,k}=(n/2\pi\mathrm{i})\oint_{C_k}(Y_k(r_k x)\,\mathrm{d}x)/(x^{n+1})\,\mathrm{d}x$, and C_k is the circle $(1/r_k)\,\mathrm{e}^{\mathrm{i}\theta},-\pi<\theta<\pi$. (Here we have implicitly assumed that $r_k>1$, but that assumption is justified later in Theorem 6.)

Lemma 5. $r_k\to 1$ *as* $k\to\infty$.

Proof. We first prove that

$$r_k=\int_0^{\infty}\frac{\mathrm{d}y}{S_k(y)}, \tag{3}$$

and then apply the Monotone Convergence Theorem. For all $k\geqslant 2$, the integral $\int_0^{\infty}\mathrm{d}y/(S_k(y))$ is convergent. We can therefore define, for $y>0$, a strictly increasing function $T_k(y)$ by $T_k(y)=\int_0^{y}(\mathrm{d}t/S_k(t))$. Let $a_k=\int_0^{\infty}(\mathrm{d}y/S_k(y))$, and for all $x\in(0,a_k)$ let $G(x)=T_k(Y_k(x))$. Then $G(0)=0$ and, by Lemma 4, $G'(x)=1$. Hence $G(x)=x$ for all x in $(0,a_k)$. Thus, T_k is a homeomorphism of $[0,\infty)$ onto $[0,a_k)$ with inverse function Y_k, and $\lim_{x\to a_k}Y_k(x)=\infty$. Y_k is a power series with positive coefficients whose smallest positive singularity is at $x=a_k$, therefore, $r_k=a_k$.

By the Monotone Convergence Theorem,

$$\lim_{k\to\infty}r_k=\int_0^{\infty}\lim_{k\to\infty}\frac{1}{S_k(y)}\,\mathrm{d}y=\int_0^{\infty}\mathrm{e}^{-y}\,\mathrm{d}y=1. \qquad \square$$

Theorem 6. $r_k=1+2^{-k-1}+\mathrm{o}(2^{-k})$ *as* $k\to\infty$.

Proof. Let $h=h(k,t)=e^{-kt}(1-(S_k(kt)/e^{kt}))$. By (3)

$$r_k-1=\int_0^\infty\left(\frac{1}{S_k(y)}-\frac{1}{e^y}\right)dy=I_1+I_2, \tag{4}$$

where $I_1=k\int_0^{0.8}h\,dt$ and $I_2=k\int_{0.8}^\infty h\,dt+\int_0^\infty(e^y-S_k(y))^2/(e^{2y}S_k(y))\,dy$. First, we estimate I_1, which, by Theorem 2, gives the major contribution

$$I_1=k\int_0^{0.8}e^{-kt}Z(k,t)\left(1+O\left(\frac{1}{k}\right)\right)dt=\frac{\sqrt{k}}{2\pi}e^k\left(1+O\left(\frac{1}{k}\right)\right)\int_0^{0.8}e^{k(-2t+\ln t)}\frac{t}{1-t}\,dt.$$

Applying the Laplace method to the last integral, we get

$$I_1=\sqrt{\frac{k}{2\pi}}e^k e^{k(-1+\ln(1/2))}\sqrt{\frac{2\pi}{4k}}\left(1+O\left(\frac{1}{k}\right)\right)=2^{-k-1}\left(1+O\left(\frac{1}{k}\right)\right). \tag{5}$$

We still need to prove that I_2 is negligible

$$I_2=I_{2,0}+I_{2,1}+I_{2,2}, \tag{6}$$

where $I_{2,0}=k\int_{0.8}^1h\,dt$, and $I_{2,1}=k\int_1^\infty h\,dt$, and $I_{2,2}=\int_0^\infty(e^y-S_k(y))^2/(e^{2y}S_k(y))\,dy$.

Applying Theorem 3 to $I_{2,0}$, we get

$$I_{2,0}=k\sqrt{\frac{2}{\pi}}\int_{0.8}^1e^{-kt}\frac{\xi t}{1-t}Erfc(\sqrt{k}\xi)\left(1+O\left(\frac{1}{\sqrt{k}}\right)\right)dt,$$

where $\xi=|t-1-\ln t|^{1/2}$. There is an absolute constant K such that $|(\xi t/(1-t))Erfc(\sqrt{k}\xi)|\leqslant K$ for all $t\in[0.8,1]$. Thus

$$|I_{2,0}|\leqslant Kk\int_{0.8}^1e^{-kt}\,dt=O(e^{-0.8k}). \tag{7}$$

Since $S_k(t)/e^t\leqslant 1$, we also have

$$I_{2,1}\leqslant k\int_1^\infty e^{-kt}\,dt=e^{-k}=o(2^{-k}). \tag{8}$$

We likewise break up $I_{2,2}$ and apply Theorem 3

$$I_{2,2}=k\int_0^\infty\frac{(1-S_k(ky)e^{-ky})^2}{S_k(ky)}\,dy=I'_{2,2}+I''_{2,2}+I'''_{2,2}, \tag{9}$$

where $I'_{2,2}=k\int_0^{0.8}$, and $I''_{2,2}=k\int_{0.8}^1$, and $I'''_{2,2}=k\int_1^\infty$. Using Theorem 2, and then the fact that

$$\left|\sqrt{\frac{2}{\pi}}\frac{y\xi(y)}{y-1}Erfc(\xi(y)\sqrt{k})\right|\leqslant\frac{1}{2} \tag{10}$$

for all $y \in [0,1)$ and all k, we get

$$I'_{2,2} = \mathrm{O}\left(\mathrm{e}^{2k} \int_0^{0.8} (y^2\mathrm{e}^{-3y})^k \,\mathrm{d}y\right) = \mathrm{o}(2^{-k}). \tag{11}$$

Using Szegő's theorem, (10), and the fact that $S_k(ky)/\mathrm{e}^{ky} \leqslant 1$ we get

$$I''_{2,2} \leqslant 2k \int_{0.8}^1 \mathrm{e}^{-ky}\,\mathrm{d}y = \mathrm{O}(k\mathrm{e}^{-0.8k}) = \mathrm{o}(2^{-k}).$$

For $I'''_{2,2}$ the estimates are similar, except now we use that fact that $Erfc(x) > c\mathrm{e}^{-x^2}/(x+1)$ for some absolute positive constant c [3]. Hence

$$I'''_{2,2} = \mathrm{O}\left(k \int_1^\infty \mathrm{e}^{-ky}\,\mathrm{e}^{k\xi(y)^2}(\sqrt{k}\xi(y)+1)\frac{y-1}{y\xi(y)}\,\mathrm{d}y\right)$$

$$= \mathrm{O}\left(k^{3/2} \int_1^\infty \mathrm{e}^{-ky}\mathrm{e}^{k(y-1-\ln y)}\,\mathrm{d}y\right) = \mathrm{o}(2^{-k}).$$

Combining all these estimates we get $r_k - 1 = 2^{-k-1} + \mathrm{o}(2^{-k})$ as was to be shown. □

We have estimated the radius of convergence r_k. It is clear from (2) that we also need to know something about how Y_k behaves near $x = r_k$. To simplify notation, let $\eta = \eta_k = Y_k(1)$. Recall that $T_k(y)$ is the inverse of Y_k so that

$$T_k(\eta) = \int_0^\eta \frac{\mathrm{d}t}{S_k(t)} = 1. \tag{12}$$

A preliminary bound for η is the following lemma.

Lemma 7. *$\eta \leqslant k$ for all sufficiently large k.*

Proof. From (12), we have

$$1 = \int_0^k \frac{\mathrm{d}t}{S_k(t)} + \int_k^\eta \frac{\mathrm{d}t}{S_k(t)}. \tag{13}$$

To prove that $\eta \leqslant k$, it suffices to prove that the second of the two integrals on the right is negative. The first of the two integrals in (13) is decomposed as

$$\int_0^k \frac{\mathrm{d}t}{S_k(t)} = k\int_0^{0.8} \frac{\mathrm{d}t}{S_k(kt)} + k\int_{0.8}^1 \frac{\mathrm{d}t}{S_k(kt)}. \tag{14}$$

The first integral in (14) is easy to estimate using Szegő's theorem:

$$k\int_0^{0.8} \frac{\mathrm{d}t}{S_k(kt)} = k\int_0^{0.8} \mathrm{e}^{-kt}\,\mathrm{d}t + \left(\frac{k\mathrm{e}^k}{\sqrt{2\pi k}} \int_0^{0.8} (t\mathrm{e}^{-2t})^k \left(\frac{t}{1-t}\right)\mathrm{d}t\right)\left(1+\mathrm{O}\left(\frac{1}{k}\right)\right).$$

Applying Laplace's method, we get

$$k\int_0^{0.8}\frac{\mathrm{d}t}{S_k(kt)} = 1-\mathrm{e}^{-0.8k}+\frac{k\mathrm{e}^k}{\sqrt{2\pi k}}\left(\frac{1}{2e}\right)^k\frac{\sqrt{2\pi}}{\sqrt{4k}}\left(1+\mathrm{O}\left(\frac{1}{k}\right)\right)$$

$$= 1+2^{-k-1}+\mathrm{o}(2^{-k}). \tag{15}$$

By similar arguments, one can verify that

$$k\int_{0.8}^{1}\frac{\mathrm{d}t}{S_k(kt)} = \mathrm{o}(2^{-k}). \tag{16}$$

Putting (15) and (16) back into (14), we get

$$\int_k^{\eta}\frac{\mathrm{d}t}{S_k(t)} = -\,2^{-k-1}+\mathrm{o}(2^{-k})<0. \tag{17}$$

Since the integrand is positive, it follows that $\eta \leqslant k$. □

A simple lower bound for η is needed as well.

Lemma 8. $\liminf_{k\to\infty}\eta/k > \frac{1}{2}$.

Proof. Putting $S_k(y) < \mathrm{e}^y$ into (17), we get

$$\mathrm{e}^{-\eta}-\mathrm{e}^{-k} = \int_\eta^k \mathrm{e}^{-t}\,\mathrm{d}t \leqslant \int_\eta^k\frac{\mathrm{d}t}{S_k(y)} = 2^{-k-1}+\mathrm{o}(2^{-k}).$$

Hence

$$\eta > \ln\left(2^{k+1}+\mathrm{o}(2^k)\right) = k\ln 2+\mathrm{o}(k). \qquad \square$$

This lower bound can now be used to improve our upper bound.

Lemma 9. $\limsup_{k\to\infty}\eta/k < 1$.

Proof. From Lemma 7, we know $\limsup_{k\to\infty}\eta/k \leqslant 1$. If the inequality is not strict, then there must exist as sequence $k_1<k_2<k_3<\cdots$ such that $\lim_{i\to\infty}\eta_{k_i}/k_i = 1$. Putting this into (12), we get

$$1 = \lim_{i\to\infty} k_i\int_0^1\frac{\mathrm{d}t}{S_{k_i}(k_i t)}.$$

But from our estimates for r_k, we know that

$$k_i\int_0^1\frac{\mathrm{d}t}{S_{k_i}(k_i t)} = 1+2^{-k_i-1}+\mathrm{o}(2^{-k_i}).$$

The contradiction $2^{-k_i-1} = \mathrm{o}(2^{-k_i})$ establishes the lemma. □

With these lemmas at our disposal, we can now prove the following theorem.

Theorem 10. *$Y_k(1) = (k+1)\ln 2 + O(1/k)$ as $k \to \infty$.*

Proof. We again use

$$1 = \int_0^\eta \frac{dt}{S_k(t)} = k \int_0^{\eta/k} \frac{dt}{S_k(kt)}.$$

By Lemma 9, there is an $\varepsilon > 0$ such that $\eta/k < 1 - \varepsilon$ for all large k. We can, therefore, apply Szego's approximations to the integral above to get

$$1 = 1 - e^{-\eta} + \frac{ke^k}{\sqrt{2\pi k}} \left(\int_0^{\eta/k} e^{k(-2t+\ln t)} \frac{t}{1-t} dt \right) \left(1 + O\left(\frac{1}{k}\right)\right). \tag{18}$$

Applying the method of Laplace to (18), we get

$$1 = 1 - e^{-\eta} + \frac{ke^k}{\sqrt{2\pi k}} e^{k(-1-\ln 2)} \frac{\sqrt{2\pi}}{\sqrt{4k}} \left(1 + O\left(\frac{1}{k}\right)\right).$$

Taking logarithms, we get

$$\eta = (k+1)\ln 2 + O\left(\frac{1}{k}\right). \qquad \square$$

3. Convergence

Since $r_k \to 1$, we expect that, as $k \to \infty$, $Y_k(r_k x)$ converges to $Y(x)$, and

$$J_{n,k} \sim \frac{n}{2\pi i} \int_{C_k} \frac{Y(x)\,dx}{x^{n+1}} dx = 1. \tag{19}$$

However, this needs to be carefully justified. Although Y_k has an algebraic singularity for every k, the singularity becomes progressively weaker as k increases. The limit function $Y(x)$ has a logarithmic singularity at $x = 1$, so $Y_k(r_k x)$ and $Y(x)$ could conceivably behave quite differently near $x = 1$. Hence, we need to carefully study the convergence of Y_k to Y.

To fix some notation, we first introduce some regions and curves that will be referred to repeatedly. Let $T(\xi) = 1 - e^{-\xi}$, so we have

$$\xi = \log\left(\frac{1}{1-x}\right) \quad \Leftrightarrow \quad T(\xi) = x. \tag{20}$$

A region of the ξ plane in which T is conformal is

$$D_1 = \{\xi: |\Im m(\xi)| < \pi\}. \tag{21}$$

Let D_2 be the region of the ξ plane that contains the origin and is bounded by the following curve:

$$\Re e(\xi) = \log\left(\frac{1}{2|\sin(\theta/2)|}\right), \tag{22}$$

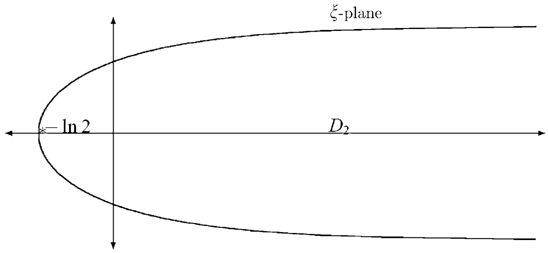

Fig. 1. Image of the unit circle under $\log(1/(1-x))$.

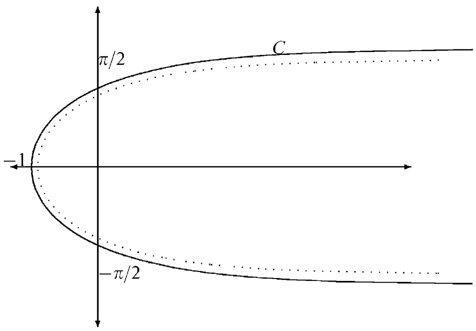

Fig. 2. The contour C in the ξ-plane.

$$\Im m(\xi) = \frac{\pi}{2} - \frac{\theta}{2}, \tag{23}$$

where $0 \leqslant \theta < 2\pi$. D_2 is an unbounded region with two horizontal asymptotes: $\Im m(\xi) = \pi/2$ and $\Im m(\xi) = -\pi/2$ (see Fig. 1).

The image of D_2 under T is the open unit disk $|x| < 1$ in the x plane. Any small neighborhood of 1 in the unit disk is mapped by T^{-1} onto a neighborhood of ∞ in D_2. D_2 is properly contained in a region $\mathscr{P}$ where none of the S_k's have a root.

Lemma 11. *S_k has no zeros in* $\mathscr{P} = \{\xi : \xi = s + \mathrm{i}t,\ t^2 \leqslant 4(s+1), s > -1\}$.

Proof. See [29, p. 13,18] (set $v = 0$ in Corollary 1.3). □

Choose an unbounded contour C in $D_1 \cap \mathscr{P}$ as shown in Fig. 2. The contour C starts at $\infty + ((\pi/2) + \delta)\mathrm{i}$ and winds around D_2 counterclockwise so that it returns to $\infty - ((\pi/2) + \delta)\mathrm{i}$. We can choose C so that it runs parallel to real axis for the two parts of the curve that have real part larger than x_0, where x_0 is a large but fixed positive constant.

Lemma 12. *There is a k_0 and an x_0 such that, for all $k \geqslant k_0$ and all $w \in C$ with $\Re e(w) \geqslant x_0$, we have $|T_k(w)| > 1$. Furthermore, $T_k(w) \to T(w)$ uniformly for all $w \in C$ with $\Re e(w) \geqslant x_0$.*

The proof of Lemma 12 is a bit tedious because there are many cases to examine. To prevent it from overwhelming the rest of the paper we have put the proof of this lemma in an appendix. For now the reader may wish to assume Lemma 12, and proceed directly to Theorem 13.

Theorem 13. *There is a k_0 such that, for all $k \geqslant k_0$, Y_k is conformal on the unit disk $|x| < 1$. Furthermore $Y_k \to Y$ uniformly on compact subsets of this disk.*

Proof. Let $R(C)$ be the open region that is bounded by C. Since $S_k(\xi) \to e^{\xi}$ uniformly on compact sets, we have $T_k(\xi) \to T(\xi)$ uniformly for all ξ in C with $\Re e(\xi) \leqslant x_0$. We can therefore choose k_0 and a small positive constant δ such that $|T_k(\xi)| > 1 + \delta/2$ for all ξ in C with real part less than or equal to x_0. This plus Lemma 12 imply that

- $|T_k(\xi)| > 1$ for all $k \geqslant k_0$ and all $\xi \in C$.
- $T_k(x) \to T(x)$ uniformly for all ξ in C. (24)

Since T is conformal, we have for $|x| < 1$,

$$1 = \frac{1}{2\pi i} \oint_C \frac{T'(z)}{T(z) - x} \, dz. \tag{25}$$

The solution to $T(\xi) = x$ is given by the residue theorem as

$$Y(x) = \frac{1}{2\pi i} \oint_C z \frac{T'(z)}{T(z) - x} \, dz. \tag{26}$$

On the other hand, we know

$$\oint_C \frac{T_k'(z)}{T_k(z) - x} \, dz \to \frac{1}{2\pi i} \oint_C \frac{T'(z)}{T(z) - x} \, dz$$

uniformly on compact subsets of $|x| < 1$, and both integrals are non-negative integers. From (24) and (25), we may conclude that, for $k \geqslant k_0$ and for all x such that $|x| < 1$,

$$1 = \frac{1}{2\pi i} \oint_C \frac{T_k'(z) \, dz}{T_k(z) - x}.$$

Thus the whole open unit disk is mapped in a 1–1 fashion by T_k. The equation $T_k(z) = x$ has a *unique* solution $\hat{Y}_k(x)$. By the residue theorem,

$$\hat{Y}_k(x) = \frac{1}{2\pi i} \oint_C z \frac{T_k'(z)}{T_k(z) - x} \, dz. \tag{27}$$

We know that the right side of (27) is well defined and analytic for $|x| < 1$ since $|T_k| > 1$. We must therefore have $\hat{Y}_k(x) = Y_k(x)$ for all x in a neighborhood of zero. But then, by the identity theorem, $\hat{Y}_k(x) = Y_k(x)$ for all x in $|x| < r_k$. There is a k_0 such that T_k is conformal in $|x| < 1$ for all $k \geqslant k_0$.

Using (27) and the fact that $T_k \to T$ uniformly on C, we can now conclude that $Y_k \to Y$ uniformly on compact subsets of $|x| < 1$. □

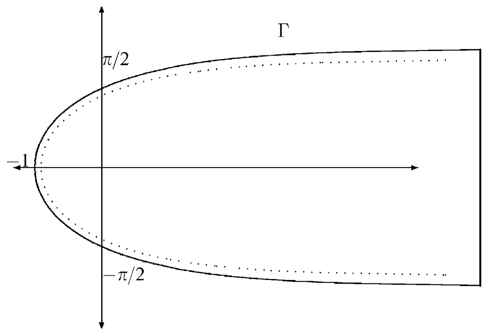

Fig. 3. The contour Γ.

For any $r > 1$ and any $\varepsilon > 0$, let

$$\mathscr{D}_{r,\varepsilon} = \{z : |z| < r \text{ and } |z-1| > \varepsilon\}.$$

Then we have the following theorem.

Theorem 14. *For every $\varepsilon > 0$, there is an $r > 1$ and a k_0 such that, for all $k > k_0$, $Y_k(z)$ can be analytically continued into $D_{r,\varepsilon}$. Furthermore, $Y_k \to Y$ uniformly on $D_{r,\varepsilon}$.*

Proof. For any $\varepsilon > 0$, $T^{-1}(\bar{D}_{1,\varepsilon/2})$ is a compact region of the ξ-plane. Choose a closed contour Γ in $D_1 \cap P$ in such a way that its right boundary extends that of $T^{-1}(\overline{D_{1,\varepsilon/2}})$ and $T^{-1}(\overline{D_{1,\varepsilon/2}})$ lies in the region enclosed by Γ (see Fig. 3). Recall $\mathscr{R}(\Gamma)$ denotes the open region bounded by Γ, and that $T(\Gamma)$ encloses a region that is slightly larger than $D_{1,\varepsilon/2}$. So

$$T_k(\xi) = \int_0^\xi \frac{\mathrm{d}y}{S_k(y)} \to \int_0^\xi \frac{\mathrm{d}y}{\mathrm{e}^y} = T(y) \tag{28}$$

uniformly for all $\xi \in \overline{\mathscr{R}(\Gamma)}$, where the integration path is a straight line segment from 0 to ξ. It follows that the contour $T_k(\Gamma)$ becomes arbitrarily close to $T(\Gamma)$, provided k is large enough. Hence, for k sufficiently large, there is an $r > 1$ and $\delta(r,\varepsilon) > 0$ such that

$$\text{Distance}(\overline{D_{r,\varepsilon}}, T_k(\Gamma)) \geqslant \delta(r,\varepsilon) > 0. \tag{29}$$

Given an arbitrary $x \in D_{r,\varepsilon}$, we have $T^{-1}(x) \in \mathscr{R}(\Gamma)$, i.e., $Y(x) \in \mathscr{R}(\Gamma)$. Thus, Y is a solution of $T(y) = x$, and by the residue theorem

$$Y(x) = \frac{1}{2\pi \mathrm{i}} \oint_\Gamma \xi \frac{T'(\xi)}{T(\xi) - x} \, \mathrm{d}\xi. \tag{30}$$

By (29), the integral $(1/2\pi\mathrm{i}) \oint_\Gamma ((\xi T_k'(\xi))/(T_k(\xi) - x)) \, \mathrm{d}\xi$ is well defined for all k larger than a constant k_0. Hence we can define, for $x \in D_{r,\varepsilon}$ and $k \geqslant k_0$,

$$\tilde{Y}_k(x) = \frac{1}{2\pi \mathrm{i}} \oint_\Gamma \xi \frac{T_k'(\xi)}{T_k(\xi) - x} \, \mathrm{d}\xi. \tag{31}$$

We have not yet shown any relationship between $\tilde{Y}_k$ and Y_k. By (28) and (30), we have

$$\tilde{Y}_k(x) \to Y(x) \tag{32}$$

uniformly for $x \in D_{r,\varepsilon}$. To show that $\tilde{Y}_k$ is an analytic continuation of Y_k, we begin by observing that Y_k is conformal in a neighborhood $\mathcal{N}$ of $x=0$ since $Y_k'(0)=1\neq 0$. From Lemma 4, we have $x=T_k(y)$ for x in a neighborhood of $x=0$. Thus

$$Y_k \text{ is the unique solution to } x=T_k(y). \tag{33}$$

On the other hand, from (31), $\tilde{Y}_k$ is also the unique root of $T_k(y)=x$. Thus, $Y_k=\tilde{Y}_k$ for all $x\in\mathcal{N}$. It follows, by the identity theorem that $Y_k=\tilde{Y}_k$ for all x in $D_{r,\varepsilon}\cap\{x:|x|<r_k\}$. Even though the radius of convergence of Y_k is less than r, we have an analytic continuation $\tilde{Y}_k$ to $\overline{D_{r,\varepsilon}}$. □

4. Final estimations

Recall that $J_{n,k}=(n/2\pi\mathrm{i})\oint_{C_k}((Y_k(r_kx))/(x^{n+1}))\,\mathrm{d}x$. Our goal is to show that, for fixed d and $k=\lfloor\log_2 n\rfloor+d$, we have $J_{n,k}=1+\mathrm{o}(1)$ as $n\to\infty$. An equivalent way to state this is

$$\frac{1}{2\pi\mathrm{i}}\oint_{C_k}\frac{Y_k(r_kx)-Y(x)}{x^{n+1}}\,\mathrm{d}x=\mathrm{o}\left(\frac{1}{n}\right), \tag{34}$$

where C_k is the contour $r_k^{-1}\mathrm{e}^{\mathrm{i}\theta}$, $-\pi\leqslant\theta\leqslant\pi$. Let $C_{k,2}=C_k\cap D_{r,\varepsilon}$ be the part of the contour that lies in $D_{r,\varepsilon}$, and let $C_{k,1}$ be the remainder. We split the integral as follows:

$$\frac{1}{2\pi\mathrm{i}}\oint_{C_k}\frac{Y_k(r_kx)-Y(x)}{x^{n+1}}\,\mathrm{d}x=\int_{C_{k,1}}(*)+\int_{C_{k,2}}(*)=S_{k,1}+S_{k,2}. \tag{35}$$

Lemma 15. $S_{k,2}=\mathrm{o}(1/n)$.

Proof. Integrating by parts we get

$$S_{k,2}=\delta\left(\frac{1}{2\pi\mathrm{i}}\frac{Y_k(r_kx)-Y(x)}{(-n)x^n}\right)+\frac{1}{2\pi\mathrm{i}}\int_{C_{k,2}}\frac{Y_k'(r_kx)-Y'(x)}{nx^n}\,\mathrm{d}x=S_{k,2}'+S_{k,2}'', \tag{36}$$

where δ denotes the difference over the appropriate endpoints of the contour $C_{k,2}$. Adding and subtracting, we get $Y_k(r_kx)-Y(x)=(Y_k(r_kx)-Y(r_kx))+(Y(r_kx)-Y(x))$. If x is an endpoint of $C_{k,2}$, then $r_kx\in D_{r,\varepsilon}$ and, by Theorem 14, $(Y_k(r_kx)-Y(r_kx))\to 0$ as $k\to\infty$. Similarly, $Y(r_kx)-Y(x)\to 0$, so we have $|S_{k,2}'|\leqslant(1/n)r_k^n\mathrm{o}(1)=\mathrm{o}(1/n)$. To estimate $S_{k,2}''$, note that since $Y_k(r_kx)\to Y(x)$ uniformly in $D_{r,\varepsilon}$, its derivative does as well, i.e., $Y_k'(r_kx)r_k\to Y'(x)$ uniformly in $D_{r,\varepsilon}$. □

To take care of the remaining term of (35), we will prove that $S_{k,1}=\mathrm{o}(1/n)$. However, first we require two lemmas that are needed for the proof. Let $\varepsilon_k(y)$ be the function of y defined by

$$T_k(y)=T(y)+\varepsilon_k(y/k). \tag{37}$$

Lemma 16. *For any* $\varepsilon>0$, $\varepsilon_k((1/k)Y_k(\mathrm{e}^{\mathrm{i}\theta}))=\mathrm{O}(\sqrt{\log n}/n)$, *uniformly for* $-\varepsilon\leqslant\theta\leqslant\varepsilon$.

Proof. If $y \in \{Y_k(x): x \in C_{k,1}\}$, then $|y/k| \leqslant (Y_k(1)/k) = \ln 2 + O(1/k) < 1$. By Caratheodory's theorem [11], the image of the closed unit disk under Y_k has a bounded imaginary part; for any $\delta > 0$ there is a k_0 such that, for all $k \geqslant k_0$, $\sup_{|x| \leqslant 1} |\Im m(Y_k(x))| \leqslant (\pi/2) + \delta$. Thus $\{(Y_k(x)/k): x \in C_{1,k}\}$ is contained in the rectangle R_k with vertices $\pm(\mathrm{i}((\pi/2)+\delta)/k)$, $\ln 2 \pm (\mathrm{i}((\pi/2)+\delta)/k)$. If k is large, obviously R_k is contained in the domain $\{z: |z\mathrm{e}^{1-z}| \leqslant 1,\ |z| \leqslant 1\}$. By Szegő's approximation, for all $y \in \{Y_k(x): x \in C_{k,1}\}$

$$\begin{aligned} T_k(y) &= k \int_0^{y/k} \frac{\mathrm{d}t}{\mathrm{e}^{kt}(1 - Z(k,t)(1 + O(1/k)))} \\ &= k \int_0^{y/k} \mathrm{e}^{-kt} \left(1 + Z(k,t)\left(1 + O\left(\frac{1}{k}\right)\right)\right) \mathrm{d}t \\ &= 1 - \mathrm{e}^{-y} + \sqrt{\frac{k}{2\pi}} \int_0^{y/k} \left(\frac{t}{1-t}\right) (t\mathrm{e}^{1-2t})^k \left(1 + O\left(\frac{1}{k}\right)\right) \mathrm{d}t \end{aligned} \tag{38}$$

uniformly for $y \in \{Y_k(x): x \in C_{k,1}\}$. From (37) and (38), we get

$$\varepsilon_k(Y_k(\mathrm{e}^{\mathrm{i}\theta}/k)) = \sqrt{\frac{k}{2\pi}} \int_0^{Y_k(\mathrm{e}^{\mathrm{i}\theta})/k} \left(\frac{t}{1-t}\right) (t\mathrm{e}^{1-2t})^k \left(1 + O\left(\frac{1}{k}\right)\right) \mathrm{d}t. \tag{39}$$

By the maximum modulus principle and calculus, $\max_{x \in R_k} |x\mathrm{e}^{1-2x}| \leqslant \frac{1}{2} + O(1/k^2)$. Putting this into (39), we get $|\varepsilon((Y_k(\mathrm{e}^{\mathrm{i}\theta})/k))| = O(\sqrt{k}2^{-k})$. If d is fixed, and $k = \log_2 n + d$, then this is $O(\sqrt{\log n}/n)$. □

Lemma 17. *There is a positive constant $\beta > 0.084$ such that, for any $\varepsilon > 0$, we have*

$$\varepsilon_k' \left(\frac{1}{k} \log\left(\frac{1}{1 - \xi_k(\mathrm{e}^{\mathrm{i}\theta})}\right)\right) \frac{\xi_k'(\mathrm{e}^{\mathrm{i}\theta}) r_k}{k(1 - \xi_k(\mathrm{e}^{\mathrm{i}\theta}))} = O\left(\frac{1}{n^\beta}\right)$$

uniformly for $-\varepsilon \leqslant \theta \leqslant \varepsilon$.

Proof. Differentiating (37) with respect to y, we get $(1/S_k(y)) = \mathrm{e}^{-y} + (1/k)\varepsilon_k'(1/ky)$. In particular, with $y = Y_k(\mathrm{e}^{\mathrm{i}\theta})$, we have

$$\frac{1}{k} \varepsilon_k' \left(\frac{1}{k} Y_k(\mathrm{e}^{\mathrm{i}\theta})\right) = \frac{1 - (S_k(Y_k(\mathrm{e}^{\mathrm{i}\theta}))/(\mathrm{e}^{Y_k(\mathrm{e}^{\mathrm{i}\theta})}))}{S_k(Y_k(\mathrm{e}^{\mathrm{i}\theta}))}. \tag{40}$$

If we define $\xi_k(x) = T(Y_k(x))$ for $|x| \leqslant 1$, then $Y_k(x) = \log(1/(1 - \xi_k(x)))$ and we have the functional equation

$$\xi_k(x) = x - \varepsilon_k \left(\frac{1}{k} \log\left(\frac{1}{1 - \xi_k(x)}\right)\right). \tag{41}$$

Since $1/(1-\xi_k(\mathrm{e}^{\mathrm{i}\theta})) = \mathrm{e}^{Y_k(\mathrm{e}^{\mathrm{i}\theta})}$, we have

$$\varepsilon_k'\left(\frac{1}{k}\log\left(\frac{1}{1-\xi_k(\mathrm{e}^{\mathrm{i}\theta})}\right)\right)\frac{1}{k(1-\xi_k(\mathrm{e}^{\mathrm{i}\theta}))} = \frac{1-S_k(Y_k(\mathrm{e}^{\mathrm{i}\theta}))/\mathrm{e}^{Y_k(\mathrm{e}^{\mathrm{i}\theta})}}{S_k(Y_k(\mathrm{e}^{\mathrm{i}\theta}))/\mathrm{e}^{Y_k(\mathrm{e}^{\mathrm{i}\theta})}}. \tag{42}$$

To simplify (42), we again use Szegő's approximation. If we let $t = Y_k(\mathrm{e}^{\mathrm{i}\theta})/k$, then

$$\frac{S_k(Y_k(\mathrm{e}^{\mathrm{i}\theta}))}{\mathrm{e}^{Y_k(\mathrm{e}^{\mathrm{i}\theta})}} = 1 - Z(k,t)\left(1+\mathrm{O}\left(\frac{1}{k}\right)\right).$$

Let $b = \max_{t\in R_k}|t\mathrm{e}^{1-t}| = \max_{t\in\partial R_k}|t\mathrm{e}^{1-t}|$, so that

$$1 - \frac{S_k(Y_k(\mathrm{e}^{\mathrm{i}\theta}))}{\mathrm{e}^{Y_k(\mathrm{e}^{\mathrm{i}\theta})}} = \mathrm{O}\left(\frac{b^k}{\sqrt{k}}\right).$$

There is a k_0 such that $b < 0.943$ for all $k > k_0$. Let $\beta = (-\ln(b)/\ln 2) > 0.0846\ldots$. If $k = \lfloor \mu_n \rfloor + d$, then for all sufficiently large n we have

$$b^k = n^{(\ln b/\ln 2)} + \mathrm{O}(1/\ln n) < \frac{1}{n^\beta}.$$

Hence

$$\left|\varepsilon_k'\left(\frac{1}{k}\log\left(\frac{1}{1-\xi_k(\mathrm{e}^{\mathrm{i}\theta})}\right)\right)\frac{1}{k(1-\xi_k(\mathrm{e}^{\mathrm{i}\theta}))}\right| = \mathrm{O}(b^k) = \mathrm{O}\left(\frac{1}{n^\beta}\right). \tag{43}$$

Differentiating both sides of the functional equation (41), we get

$$\xi_k'(x) + \varepsilon_k'\left(\frac{1}{k}\log\frac{1}{1-\xi_k(x)}\right)\frac{\xi_k'(x)}{k(1-\xi_k(x))} = 1.$$

It follows from this and (43)

$$\xi_k'(\mathrm{e}^{\mathrm{i}\theta}) = \frac{1}{1+\varepsilon_k'((1/k)\log(1/(1-\xi_k(\mathrm{e}^{\mathrm{i}\theta}))))(1/(k(1-\xi_k(\mathrm{e}^{\mathrm{i}\theta}))))} = \mathrm{O}(1). \tag{44}$$

The lemma now follows directly from (44) and (43). □

With these two lemmas at our disposal, we can proceed with the proof that $S_{k,1}$ is negligible in (35). Using $|r_k x| = 1$, we write

$$nS_{k,1} = \frac{n}{2\pi\mathrm{i}}\int_{C_{k,1}} \frac{Y_k(r_k x) - Y(x)}{x^{n+1}}\,\mathrm{d}x = \frac{n}{2\pi\mathrm{i}}\int_{C_{k,1}} \log\left(\frac{1-x}{1-\xi_k(r_k x)}\right)\frac{\mathrm{d}x}{x^{n+1}}$$

$$= \frac{n}{2\pi\mathrm{i}}\int_{C_{k,1}} \log\left(\frac{1-x}{1-r_k x+\varepsilon_k((1/k)\log(1/(1-\xi_k(r_k x))))}\right)\frac{\mathrm{d}x}{x^{n+1}}.$$

Integrating by parts, we get

$$nS_{k,1} = \frac{-1}{2\pi\mathrm{i}}S_{k,1}' + S_{k,1}'' \tag{45}$$

where

$$S'_{k,1} = \frac{1}{x^n}\log\left(\frac{1-x}{1-r_kx+\varepsilon_k((1/k)\log(1/(1-\xi_k(r_kx))))}\right)\Bigg|_{\partial C_{k,1}}$$

and

$$S''_{k,1} = \int_{C_{k,1}} \frac{1}{nx^n}\left(\frac{-1}{1-x}+\frac{r_k-\varepsilon'_k}{1-r_kx+\varepsilon_k}\right)\mathrm{d}x.$$

The curve $C_{k,1}$ traces out $\{r_k^{-1}\mathrm{e}^{\mathrm{i}\theta}:\ -\alpha \leqslant \theta \leqslant \alpha\}$, where the angle of subtension α remains to be estimated. Since $|r_k^{-1}\mathrm{e}^{\mathrm{i}\alpha}-1|^2=\varepsilon^2$ we get $r_k^{-2}-2r_k^{-1}\cos\alpha+1=\varepsilon^2$. Recall that $r_k=1+2^{-k-1}+\mathrm{o}(2^{-k})$ (Theorem 6). Hence $\cos\alpha = 1-(\varepsilon^2/2)+\mathrm{o}(2^{-k})$ and consequently

$$\alpha = \varepsilon + \mathrm{o}(2^{-k}). \tag{46}$$

We, therefore, have

$$\begin{aligned} S'_{k,1} &= \mathrm{e}^{-\mathrm{i}n\alpha}r_k^n\log\left(\frac{1-r_k^{-1}\mathrm{e}^{\mathrm{i}\alpha}}{1-\mathrm{e}^{\mathrm{i}\alpha}+\varepsilon_k((1/k)\log(1/(1-\xi_k(\mathrm{e}^{\mathrm{i}\alpha}))))}\right) \\ &\quad -\mathrm{e}^{-\mathrm{i}n\alpha}r_k^n\log\left(\frac{1-r_k^{-1}\mathrm{e}^{-\mathrm{i}\alpha}}{1-\mathrm{e}^{-\mathrm{i}\alpha}+\varepsilon_k((1/k)\log(1/(1-\xi_k(\mathrm{e}^{-\mathrm{i}\alpha}))))}\right) \\ &= \mathrm{O}\left(\log\frac{1-r_k^{-1}\mathrm{e}^{\mathrm{i}\alpha}}{1-\mathrm{e}^{\mathrm{i}\alpha}+\varepsilon_k((1/k)Y_k(\mathrm{e}^{\mathrm{i}\alpha}))}\right)+\mathrm{O}\left(\log\frac{1-r_k^{-1}\mathrm{e}^{-\mathrm{i}\alpha}}{1-\mathrm{e}^{-\mathrm{i}\alpha}+\varepsilon_k((1/k)Y_k(\mathrm{e}^{-\mathrm{i}\alpha}))}\right). \end{aligned} \tag{47}$$

Using (46), that $\alpha=\varepsilon+\mathrm{o}(2^{-k})$, the fact that $r_k = 1+\mathrm{O}(2^{-k})$, and Lemma 16, we see that both of the big-O terms are o(1). Thus

$$S'_{k,1} = \mathrm{o}(1). \tag{48}$$

It remains to prove that $S''_{k,1}=\mathrm{o}(1)$. With $x=r_k^{-1}\mathrm{e}^{\mathrm{i}\theta}$, and a change of variable, we have $S''_{k,1} = (r_k^{n-1}/2\pi)(J_1+J_2)$, where

$$J_1 = \int_{-\alpha}^{\alpha} \mathrm{e}^{-(n-1)\mathrm{i}\theta}\frac{(r_k-1)-\varepsilon_k((1/k)\log(1/(1-\xi_k(\mathrm{e}^{\mathrm{i}\theta}))))}{(1-r_k^{-1}\mathrm{e}^{\mathrm{i}\theta})(1-\mathrm{e}^{\mathrm{i}\theta}+\varepsilon_k((1/k)\log(1/(1-\xi_k(\mathrm{e}^{\mathrm{i}\theta})))))}\,\mathrm{d}\theta \tag{49}$$

and

$$J_2 = -\int_{-\alpha}^{\alpha} \mathrm{e}^{-(n-1)\mathrm{i}\theta}\frac{\varepsilon'_k((1/k)\log(1/(1-\xi_k(\mathrm{e}^{\mathrm{i}\theta}))))}{1-\mathrm{e}^{\mathrm{i}\theta}+\varepsilon_k((1/k)\log(1/(1-\xi_k(\mathrm{e}^{\mathrm{i}\theta}))))}\frac{\xi'_k(\mathrm{e}^{\mathrm{i}\theta})r_k}{k(1-\xi_k(\mathrm{e}^{\mathrm{i}\theta}))}\,\mathrm{d}\theta. \tag{50}$$

We must prove that J_2 and J_1 are both o(1). To this end, decompose the integral in J_2 into three ranges:

$$J_2 = -\int_{-\alpha}^{-n^{-1+\beta/2}} - \int_{-n^{-1+\beta/2}}^{n^{-1+\beta/2}} - \int_{n^{-1+\beta/2}}^{\alpha} = -J'_2 - J''_2 - J'''_2. \tag{51}$$

If we define

$$\tilde{\varepsilon}_k(\theta)=\varepsilon_k'\left(\frac{1}{k}Y_k(\mathrm{e}^{\mathrm{i}\theta})\right)\frac{\xi_k'(\mathrm{e}^{\mathrm{i}\theta})r_k}{k(1-\xi_k(\mathrm{e}^{\mathrm{i}\theta}))}, \tag{52}$$

then

$$J_2''=\int_{-n^{\beta/2}}^{n^{\beta/2}}\mathrm{e}^{-(n-1)\mathrm{i}\theta/n}\frac{\tilde{\varepsilon}_k(\theta/n)\,\mathrm{d}\theta}{n(1-\mathrm{e}^{\mathrm{i}\theta/n}+\varepsilon_k((1/k)Y_k(\mathrm{e}^{\mathrm{i}\theta/n})))}. \tag{53}$$

Applying the mean value theorem separately to the real and imaginary parts of $\varepsilon_k((1/k)Y_k(\mathrm{e}^{\mathrm{i}\theta/n}))$, and using Lemma 17, one can verify that

$$\varepsilon_k\left(\left(\frac{1}{k}Y_k(\mathrm{e}^{\mathrm{i}\theta/n})\right)\right)=\varepsilon_k(Y_k(1)/k)+\mathrm{O}\left(\frac{1}{n^{1+\beta/2}}\right) \tag{54}$$

uniformly for $-n^{\beta/2}\leqslant\theta\leqslant n^{\beta/2}$. By (39),

$$\varepsilon_k\left(\frac{1}{k}Y_k(1)\right)=\left(\frac{\sqrt{k}\mathrm{e}^k}{\sqrt{2\pi}}\int_0^{Y_k(1)/k}\mathrm{e}^{k(-2t+\ln t)}\frac{t}{1-t}\,\mathrm{d}t\right)\left(1+\mathrm{O}\left(\frac{1}{k}\right)\right).$$

By Theorem 10 and the Laplace method, $\varepsilon_k((1/k)Y_k(1))=2^{-k-1}+\mathrm{O}((1/k)2^{-k})$, and consequently

$$\begin{aligned}\varepsilon_k\left(\frac{1}{k}Y_k(\mathrm{e}^{\mathrm{i}\theta/n})\right)&=2^{-k-1}+\mathrm{O}\left(\frac{1}{k}2^{-k}\right)+\mathrm{O}\left(\frac{1}{n^{1+\beta/2}}\right)\\&=\frac{1}{n}2^{\{\ln n/\ln 2\}-d-1}+\mathrm{O}\left(\frac{1}{n\log n}\right).\end{aligned} \tag{55}$$

Putting this back into (53), we have

$$J_2''=\int_{-n^{\beta/2}}^{n^{\beta/2}}\mathrm{e}^{-(n-1)\mathrm{i}\theta/n}\frac{\tilde{\varepsilon}_k(\theta/n)\,\mathrm{d}\theta}{n(-\mathrm{i}(\theta/n)+\mathrm{O}(n^{2\beta-2}))+2^{\{\log_2 n\}-d-1}+\mathrm{O}(1/\log n)}.$$

By (52), and Lemma 17, $\tilde{\varepsilon}_k(\theta/n)=\mathrm{O}(n^{-\beta})$ for $-n^{\beta/2}\leqslant\theta\leqslant n^{\beta/2}$. Hence

$$|J_2''|\leqslant\mathrm{O}(n^{-\beta})\int_{-n^{\beta/2}}^{n^{\beta/2}}\frac{\mathrm{d}\theta}{|\mathrm{i}\theta+2^{\{\log_2 n\}-d-1}|-\mathrm{O}(1/\log n)}=\mathrm{O}\left(\frac{\ln n}{n^\beta}\right)=\mathrm{o}(1). \tag{56}$$

For the last of the three terms in (51), we write

$$J_2'''=\frac{1}{n^{1-(\beta/2)}}\int_1^{\alpha n^{1-(\beta/2)}}\frac{\mathrm{e}^{-(n-1)\mathrm{i}\theta/n^{1-(\beta/2)}}\tilde{\varepsilon}_k(\theta/n^{1-(\beta/2)})}{(1-\mathrm{e}^{\mathrm{i}\theta/n^{1-(\beta/2))}}+\varepsilon_k(1/kY_k(\mathrm{e}^{\mathrm{i}\theta/n^{1-(\beta/2)}}))}\,\mathrm{d}\theta.$$

Since $\tilde{\varepsilon}_k(\theta/n^{1-\beta/2}) = \mathrm{O}(1/n^{\beta})$, and $\varepsilon_k((1/k)Y_k(\mathrm{e}^{\mathrm{i}\theta/n^{1-\beta/2}})) = \mathrm{O}(\sqrt{\log n}/n)$, we have

$$|J_2'''| \leqslant \mathrm{O}\left(\frac{1}{n^{\beta}}\right)\int_1^{\alpha n^{1-(\beta/2)}} \frac{1}{n^{1-\beta/2}|1-\mathrm{e}^{\mathrm{i}\theta/n^{1-\beta/2}}| - n^{1-\beta/2}\mathrm{O}(\sqrt{\log n}/n)}\,\mathrm{d}\theta$$

$$\leqslant \mathrm{O}\left(\frac{1}{n^{\beta}}\right)\int_1^{\alpha n^{1-(\beta/2)}} \frac{\mathrm{d}\theta}{2n^{1-\beta/2}\sin(\theta/2n^{1-\beta/2}) - \mathrm{o}(1)}.$$

Using the inequality $\sin\psi \geqslant 2\psi/\pi$ for $0 \leqslant \psi \leqslant \pi/2$, we get

$$|J_2'''| \leqslant \mathrm{O}\left(\frac{1}{n^{\beta}}\right)\int_1^{\alpha n^{1-\beta/2}} \frac{\mathrm{d}\theta}{(2\theta/\pi) - \mathrm{o}(1)} = \mathrm{O}\left(\frac{1}{n^{\beta}}\right)\mathrm{O}(\log n) = \mathrm{o}(1). \tag{57}$$

Similarly, $J_2' = \mathrm{o}(1)$. This completes the proof that $J_2 = \mathrm{o}(1)$. We still need to estimate J_1

$$J_1 = \int_{-\alpha}^{\alpha} \mathrm{e}^{-(n-1)\mathrm{i}\theta} \frac{-1 + r_k - \varepsilon_k((1/k)Y_k(\mathrm{e}^{\mathrm{i}\theta}))}{(1 - r_k^{-1}\mathrm{e}^{\mathrm{i}\theta})(1 - \mathrm{e}^{\mathrm{i}\theta} + \varepsilon_k((1/k)Y_k(\mathrm{e}^{\mathrm{i}\theta})))}\,\mathrm{d}\theta$$

$$= \int_{-\alpha}^{-n^{-1+\beta/2}} + \int_{n^{-1+\beta/2}}^{\alpha} + \int_{-n^{-1+\beta/2}}^{n^{-1+\beta/2}} = J_1' + J_1'' + J_1'''. \tag{58}$$

After a change of variable, J_1''' becomes

$$J_1''' = \int_{-n^{\beta/2}}^{n^{\beta/2}} \mathrm{e}^{-(n-1)\mathrm{i}\theta/n} \frac{[(r_k - 1) - \varepsilon_k((1/k)Y_k(\mathrm{e}^{\mathrm{i}\theta/n}))]n}{[n(1 - r_k^{-1}\mathrm{e}^{\mathrm{i}\theta/n})][n(1 - \mathrm{e}^{\mathrm{i}\theta/n} + \varepsilon_k((1/k)Y_k(\mathrm{e}^{\mathrm{i}\theta/n})))]}\,\mathrm{d}\theta. \tag{59}$$

The numerator of the integrand is uniformly o(1) by (55) and Theorem 6:

$$n((r_k - 1) - \varepsilon_k((1/k)Y_k(\mathrm{e}^{\mathrm{i}\theta/n}))) = n[2^{-k-1} + \mathrm{o}(2^{-k}) - 2^{-k-1} + \mathrm{O}(1/n\ln n)] = \mathrm{o}(1). \tag{60}$$

The first factor in the denominator of (59) is

$$|n(1 - r_k^{-1}\mathrm{e}^{\mathrm{i}\theta/n})| = |2^{\{\log_2 n\}-d-1} - \mathrm{i}\theta + \mathrm{o}(1)| \geqslant |2^{-d-1} - \mathrm{i}\theta + \mathrm{o}(1)|.$$

Similarly, the second factor in the denominator of (59) has magnitude

$$|n(1 - \mathrm{e}^{\mathrm{i}\theta/n} + \varepsilon_k((1/k)Y_k(\mathrm{e}^{\mathrm{i}\theta/n})))| \geqslant |2^{-d-1} - \mathrm{i}\theta + \mathrm{o}(1)|.$$

Thus

$$|J_1'''| \leqslant \mathrm{o}(1)\int_{-n^{\beta/2}}^{n^{\beta/2}} \frac{\mathrm{d}\theta}{|2^{-d-1} - \mathrm{i}\theta + \mathrm{o}(1)|^2} = \mathrm{o}(1).$$

Similarly,

$$J_1'' = \int_1^{\alpha n^{1-\beta/2}} \frac{\mathrm{e}^{-(n-1)\mathrm{i}\theta/n^{1-\beta/2}} n^{1-\beta/2}[(r_k - 1) - \varepsilon_k((1/k)Y_k(\mathrm{e}^{\mathrm{i}\theta/n^{1-\beta/2}}))]\,\mathrm{d}\theta}{n^{1-\beta/2}(1 - r_k^{-1}\mathrm{e}^{\mathrm{i}\theta/n^{1-\beta/2}})n^{1-\beta/2}(1 - \mathrm{e}^{\mathrm{i}\theta/n^{1-\beta/2}} + \varepsilon_k((1/k)Y_k(\mathrm{e}^{\mathrm{i}\theta/n^{1-\beta/2}})))}.$$

By our previous estimates, $|\varepsilon_k((1/k)Y_k(e^{i\theta/n^{1-\beta/2}}))| = O(\sqrt{\ln n}/n)$ uniformly. Hence, in the denominator we have

$$\left| n^{1-\beta/2}\left(1 - e^{i\theta/n^{1-\beta/2}} + \varepsilon_k\left(\frac{1}{k}Y_k(e^{i\theta/n^{1-\beta/2}})\right)\right)\right| \geqslant \frac{2\theta}{\pi} - 0.1.$$

For the first factor in the denominator note that

$$|n^{1-\beta/2}(1 - r_k^{-1}e^{i\theta/n^{1-\beta/2}})| = n^{1-\beta/2}\sqrt{r_k^{-2} + 1 - 2r_k^{-1}\cos\frac{\theta}{n^{1-\beta/2}}}.$$

But we know $\theta/(n^{1-\beta/2}) \leqslant \alpha = \varepsilon + o(2^{-k})$. Therefore, for ε sufficiently small, we have $\cos\theta/(n^{1-\beta/2}) \leqslant 1 - \frac{1}{3}(\theta/n^{1-\beta/2})^2$ for $1 \leqslant \theta \leqslant n^{1-\beta/2}\alpha$. Then, for all large n, we have

$$|n^{1-\beta/2}(1 - r_k^{-1}e^{i\theta/n^{1-\beta/2}})| \geqslant \sqrt{n^{2-\beta}(r_k-1)^2 + \frac{2}{3}r_k^{-1}\theta^2} \geqslant \frac{\theta}{\sqrt{3}}.$$

For the numerator of the integrand in J_1'', we have

$$n^{1-\beta/2}\left|(r_k - 1) - \varepsilon_k\left(\frac{1}{k}Y_k(e^{i\theta/n^{1-\beta/2}})\right)\right| \leqslant n^{1-\beta/2}|r_k - 1| + n^{1-\beta/2}\left|\varepsilon_k\left(\frac{1}{k}Y_k(e^{i\theta/n^{1-\beta/2}})\right)\right|$$

$$= O\left(\frac{1}{n^{\beta/2}}\right) + n^{1-\beta/2}O\left(\frac{\sqrt{\ln n}}{n}\right) = O\left(\frac{\sqrt{\ln n}}{n^{\beta/2}}\right).$$

Combining these estimates, we get

$$|J_2''| \leqslant O\left(\frac{\sqrt{\ln n}}{n^{\beta/2}}\right)\int_1^{\alpha n^{1-\beta/2}} \frac{d\theta}{\theta((2/\pi)\theta - 0.1)} = o(1).$$

Note that $J_1' = \overline{J_1''} = o(1)$, so $J_1 = o(1)$ too. It follows that $S_{k,1}'' = o(1)$, and therefore $S_{k,1} = o(1/n)$.

This completes the proof that $J_{n,k} = 1 + o(1)$ as $n \to \infty$. Combining this with our earlier estimates for the radius of convergence, we get

$$\begin{aligned} P_n(\Delta \leqslant k) &= r_k^{-n}J_{n,k} = \exp(-n\ln r_k)(1 + o(1)) \\ &= \exp(-n\ln(1 + 2^{-k-1} + o(2^{-k}))) + o(1). \end{aligned}$$

This proves our main result.

Theorem 18. *If* $k = \lfloor \ln n/\ln 2 \rfloor + d$ *for a fixed integer* d, *then as* $n \to \infty$,

$$P_n(\Delta \leqslant k) = \exp(-2^{\{\ln n/\ln 2\} - d - 1}) + o(1).$$

5. Discussion

One immediate consequence of Theorem 1 is that Δ is rather tightly concentrated around the mean: if $\omega(n) \to \infty$ arbitrarily slowly, then $P_n(|\Delta - \lfloor \log_2 n \rfloor| > \omega) = o(1)$. Presumably, $P_n(\Delta > c\lfloor \log_2 n \rfloor)$

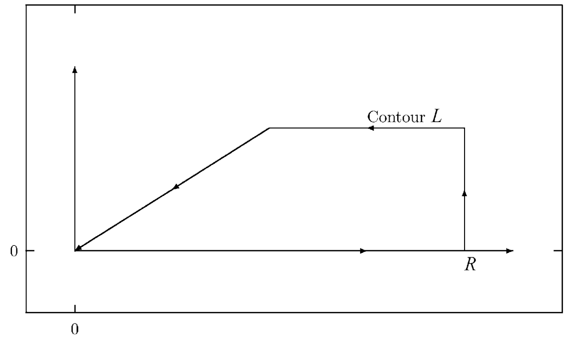

Fig. 4. The Contour L.

is quite small for $c>1$ and large n, but we have only the weak bounds that follow from Theorem 1. At the moment we do not even have a short, independent proof of the fact that the variance of Δ is $O(1)$.

We do not know how many large degree nodes there are, or how deep in the tree they appear. Let $B(T)$ be the smallest v for which vertex v has indegree Δ. Clearly, $P_n(B=v)$ is a decreasing function of v, and the root node $v=1$ is more likely than any other vertex to attain the maximum degree. Nevertheless, the mean and variance of ρ_1 are only $\ln n + O(1)$, so we must asymptotically have $P_n(B=1)=o(1)$. Presumably, the large degree vertices tend to cluster near the root, but we do not know how deep they are on average.

Appendix

Proof (*of Lemma* 12). Choose a trapezoidal contour L as shown in Fig. 4. Here $\Re e(\xi) \geqslant x_0$ and $\xi \in C$. The region that is bounded by L is contained in the zero free region for S_k, therefore $\oint_L (\mathrm{d}z/S_k(z)) = 0$. Let L_1 be vertical edge of L that traces out a line segment on the line $\Re e(\xi) = R$. Observe that $\lim_{R\to\infty} \oint_{L_1} (\mathrm{d}z/S_k(z)) = 0$. Let $A = (\pi/2) + \delta$ and assume that $x_0 \leqslant x \leqslant (\frac{1}{2} - \varepsilon)k$ for a fixed real x_0. Then

$$\int_0^{\xi} \frac{\mathrm{d}w}{S_k(w)} = \int_0^{\infty} \frac{\mathrm{d}w}{S_k(w)} - \int_x^{\infty} \frac{\mathrm{d}w}{S_k(w+\mathrm{i}A)}.$$

We partition the interval (x_0, ∞) into the following intervals:

$$(x_0, (\tfrac{1}{2}-\varepsilon)k], \qquad ((\tfrac{1}{2}-\varepsilon)k, (\tfrac{1}{2}+\varepsilon)k], \qquad ((\tfrac{1}{2}+\varepsilon)k, 0.8k], \qquad (0.8k, \infty).$$

The first case to consider is $x \in (x_0, ((1/2)-\varepsilon)k]$. Splitting up the integral, we get

$$T_k(\xi) = \int_0^{\xi} \frac{\mathrm{d}w}{S_k(w)} = r_k - \int_x^{\infty} \frac{\mathrm{d}w}{S_k(w+\mathrm{i}A)} = r_k - \pi_1 - \pi_2 - \pi_3, \tag{A.1}$$

where

$$\pi_1 = \int_x^{((1/2)-\varepsilon)k} \frac{\mathrm{d}w}{S_k(w+\mathrm{i}A)}, \qquad \pi_2 = \int_{((1/2)-\varepsilon)k}^{((1/2)+\varepsilon)k}, \qquad \pi_3 = \int_{((1/2)+\varepsilon)k}^{\infty}.$$

Let $t=(w+\mathrm{i}A)/k$, and note that

$$|t| \leqslant \sqrt{(\tfrac{1}{2}-\varepsilon)^2+\tfrac{A^2}{k^2}} = (\tfrac{1}{2}-\varepsilon)+\mathrm{O}(\tfrac{1}{k^2}).$$

By Szegő's theorem

$$\pi_1 = k\int_{x/k}^{(1/2)-\varepsilon} \mathrm{e}^{-kw-\mathrm{i}A}\left(1+\frac{1}{\sqrt{2\pi k}}\frac{w+\mathrm{i}A/k}{1-(w+\mathrm{i}A/k)}\right.$$
$$\left.\left(\left(w+\frac{\mathrm{i}A}{k}\right)\mathrm{e}^{1-(w+(\mathrm{i}A/k))}\right)^k\left(1+\mathrm{O}\left(\frac{1}{k}\right)\right)\right)\mathrm{d}w$$
$$=\mathrm{e}^{-\mathrm{i}A}(\mathrm{e}^{-x}-\mathrm{e}^{-((1/2)-\varepsilon)k})$$
$$+\sqrt{k/2\pi}\,\mathrm{e}^{-2\mathrm{i}A}\mathrm{e}^k\left(1+\mathrm{O}\left(\frac{1}{k}\right)\right)\int_{x/k}^{(1/2)-\varepsilon}\frac{w+\mathrm{i}A/k}{1-w-\mathrm{i}A/k}((w+\mathrm{i}A/k)\mathrm{e}^{-2w})^k\,\mathrm{d}w.$$

Note that

$$\left|\mathrm{e}^k\int_{x/k}^{(1/2)-\varepsilon}\frac{w+\mathrm{i}A/k}{1-w-\mathrm{i}A/k}((w+\mathrm{i}A/k)\,\mathrm{e}^{-2w})^k\,\mathrm{d}w\right|$$
$$\leqslant \mathrm{e}^{-k}\left(\frac{1}{2}-\varepsilon-x/k\right)\max_{0\leqslant w\leqslant(1/2)-\varepsilon}\left|\frac{w+\mathrm{i}A/k}{1-w-\mathrm{i}A/k}\right|\left|\frac{1}{2}-\varepsilon-\mathrm{i}A/k\right|^k\mathrm{e}^{-2((1/2)-\varepsilon)k}$$
$$=\mathrm{O}(\mathrm{e}^{k(2\varepsilon+\ln((1/2)-\varepsilon))})=\mathrm{o}(2^{-k}).$$

Thus, we have

$$\pi_1=\mathrm{e}^{-\mathrm{i}A}(\mathrm{e}^{-x}-\mathrm{e}^{-((1/2)-\varepsilon)k})+\mathrm{o}(2^{-k}). \tag{A.2}$$

We likewise use Szegő's theorem to estimate π_2. Setting $t=(w+\mathrm{i}A)/k$, we have

$$\pi_2=k\mathrm{e}^{-\mathrm{i}A}\int_{(1/2)-\varepsilon}^{(1/2)+\varepsilon}\mathrm{e}^{-kw}\left(1+\frac{1}{\sqrt{2\pi k}}\frac{w+\mathrm{i}A/k}{1-(w+\mathrm{i}A/k)}\right.$$
$$\left.((w+\mathrm{i}A/k)\mathrm{e}^{1-(w+\mathrm{i}A/k)})^k\left(1+\mathrm{O}\left(\frac{1}{k}\right)\right)\right)\mathrm{d}w$$
$$=\mathrm{e}^{-\mathrm{i}A}(\mathrm{e}^{-k((1/2)-\varepsilon)}-\mathrm{e}^{-k((1/2)+\varepsilon)})$$
$$+\mathrm{e}^{-\mathrm{i}A}\sqrt{\frac{k}{2\pi}}\int_{(1/2)-\varepsilon}^{(1/2)+\varepsilon}\mathrm{e}^{-kw}\frac{w+\mathrm{i}A/k}{1-(w+\mathrm{i}A/k)}((w+\mathrm{i}A/k)\mathrm{e}^{1-(w+\mathrm{i}A/k)})^k\left(1+\mathrm{O}\left(\frac{1}{k}\right)\right)\mathrm{d}w. \tag{A.3}$$

To simplify the integral, we note that

$$\frac{w+\mathrm{i}A/k}{1-(w+\mathrm{i}A/k)}=\frac{w}{1-w}\left(1+\mathrm{O}\left(\frac{1}{k}\right)\right)$$

and

$$((w+\mathrm{i}A/k)\mathrm{e}^{1-(w+\mathrm{i}A/k)})^k=\mathrm{e}^{-\mathrm{i}A}\mathrm{e}^{\mathrm{i}A/w}\mathrm{e}^{k(1-w+\ln w)}\left(1+\mathrm{O}\left(\frac{1}{k}\right)\right).$$

By the method of Laplace

$$\int_{(1/2)-\varepsilon}^{(1/2)+\varepsilon}\frac{w}{1-w}\mathrm{e}^{k(\ln w+1-2w)}\mathrm{e}^{\mathrm{i}A/w}\,\mathrm{d}w=\mathrm{e}^{2\mathrm{i}A}\mathrm{e}^{k\ln(1/2)}\frac{\sqrt{2\pi}}{\sqrt{4k}}\left(1+\mathrm{O}\left(\frac{1}{k}\right)\right).$$

With these simplifications, (A.3) becomes

$$\pi_2=\mathrm{e}^{-\mathrm{i}A}(\mathrm{e}^{-k((1/2)-\varepsilon)}-\mathrm{e}^{-k((1/2)+\varepsilon)})+2^{-k-1}+\mathrm{O}\left(\frac{2^{-k}}{k}\right). \tag{A.4}$$

The last of the three integrals in (A.1) will be further decomposed:

$$\pi_3=\int_{((1/2)+\varepsilon)k}^{0.8k}\frac{\mathrm{d}w}{S_k(w)}+\int_{0.8k}^{\infty}\frac{\mathrm{d}w}{S_k(w)}=\pi_3'+\pi_3''. \tag{A.5}$$

Once again one can use Szegő's approximations to prove

$$\pi_3'=\mathrm{e}^{-\mathrm{i}A}(\mathrm{e}^{-((1/2)+\varepsilon)k}-\mathrm{e}^{-0.8k})+\mathrm{o}(2^{-k}). \tag{A.6}$$

To estimate π_3'', note that, for $w\geqslant 0.8k$,

$$\begin{aligned}|S_k(w+\mathrm{i}A)|&=\left|\sum_{m=0}^{k}\frac{(w+\mathrm{i}A)^m}{m!}\right|\geqslant\sum_{m=0}^{k}\frac{m((w+\mathrm{i}A)^m)}{m!}\\&=\sum_{m=1}^{k}\frac{(w^2+A^2)^{m/2}\sin(m\alpha)}{m!},\end{aligned} \tag{A.7}$$

where $\alpha=\tan^{-1}(A/w)$. Because $w\geqslant 4k/5$, we have $k\alpha\leqslant k\tan^{-1}(A/4k/5)=\frac{5}{4}A+\mathrm{O}(1/k^2)$. Hence $\sin m\alpha>0$ for $1\leqslant m\leqslant k$ and

$$\begin{aligned}|S_k(w+\mathrm{i}A)|&\geqslant(\sin\alpha)\sum_{m=1}^{k}\frac{(w^2+A^2)^{m/2}}{m!}=(\sin\alpha)(S_k(\sqrt{w^2+A^2})-1)\\&\geqslant\frac{1}{2}(\sin\alpha)S_k(\sqrt{w^2+A^2})\geqslant\frac{1}{2}(\sin\alpha)S_k(w)\geqslant\frac{1}{2}(\sin\alpha)S_{\lfloor k/2\rfloor}(w).\end{aligned}$$

For large w, $\tan^{-1}A/w\geqslant A/2w$, therefore $|\pi_3''|\leqslant 4/A\int_{0.8k}^{\infty}(w\,\mathrm{d}w/S_{\lfloor k/2\rfloor}(w))$. We can apply Szegő's theorem to $S_{\lfloor k/2\rfloor}(w)=S_{\lfloor k/2\rfloor}(\lfloor k/2\rfloor w/\lfloor k/2\rfloor)$ after observing that $w/\lfloor k/2\rfloor\geqslant 0.8k/\lfloor k/2\rfloor>1$.

Hence

$$S_{\lfloor k/2\rfloor}(w)=\mathrm{e}^{w}\left(\sqrt{\frac{2}{\pi}}\frac{\zeta(t)t}{t-1}Erfc(\sqrt{\lfloor k/2\rfloor}\zeta(t))\right)\left(1+\mathrm{O}\left(\frac{1}{\sqrt{k}}\right)\right),$$

where $t=w/\lfloor k/2\rfloor$ and $\zeta(t)=|t-1-\ln t|^{1/2}$. Notice that $t\geqslant \frac{8}{5}$, and therefore $\zeta(t)t/(t-1)\geqslant\zeta(t)$. Therefore, with a change of variable, we have

$$|\pi_3''|\leqslant\sqrt{\frac{\pi}{2}}\frac{4}{A}k^2\int_{0.8}^{\infty}\frac{\mathrm{e}^{-kw}w\,\mathrm{d}w}{\zeta(kw/\lfloor k/2\rfloor)Erfc(\sqrt{\lfloor k/2\rfloor}\zeta(kw/\lfloor k/2\rfloor))}\left(1+\mathrm{O}\left(\frac{1}{\sqrt{k}}\right)\right).$$

Notice that $\frac{1}{\zeta(kw/\lfloor k/2\rfloor)}\leqslant 1/\zeta(w)\leqslant 1/\zeta(8/5)$ for all $w\geqslant 0.8$. Using this, and the fact that $Erfc(\sigma)=\mathrm{e}^{-\sigma^2}/2\sigma(1+\mathrm{o}(1/\sigma^2))$ as $\sigma\to\infty$, we get

$$|\pi_3''|=\mathrm{O}\left(k^2\int_{0.8}^{\infty}\mathrm{e}^{-kw}w\mathrm{e}^{\lfloor k/2\rfloor\zeta^2(kw/\lfloor k/2\rfloor)}\sqrt{\lfloor k/2\rfloor}\zeta\left(\frac{kw}{\lfloor k/2\rfloor}\right)\mathrm{d}w\right). \tag{A.8}$$

Notice that $\zeta(kw/\lfloor k/2\rfloor)\leqslant k\sqrt{w}$. Putting this back into (A.8), we get

$$|\pi_3''|=\mathrm{O}\left(k^{5/2}(2e)^{-\lfloor k/2\rfloor}\int_{0.8}^{\infty}w^{-\lfloor k/2\rfloor+3/2}\,\mathrm{d}w\right)=\mathrm{o}(2^{-k}). \tag{A.9}$$

Combining our estimates for π_3' and π_3'', we get $\pi_3=\mathrm{e}^{-\mathrm{i}A}(\mathrm{e}^{-((1/2)+\varepsilon)k}-\mathrm{e}^{-0.8k})+\mathrm{o}(2^{-k})$. Putting our estimates for π_1, π_2, and π_3, back into (A.1), we get

$$T_k(\xi)=1-\mathrm{e}^{-\mathrm{i}A}(\mathrm{e}^{-x}-\mathrm{e}^{-0.8k})+\mathrm{o}(2^{-k}). \tag{A.10}$$

Since (A.10) is valid uniformly for all x in $(x_0,((1/2)-\varepsilon)k]$, we have

$$|T_k(\xi)|\geqslant\sqrt{1-2\mathrm{e}^{-x}\cos A+\mathrm{e}^{-2x}}-\mathrm{e}^{-0.8k}-2^{-k}. \tag{A.11}$$

Because $A=(\pi/2)+\delta$, we have $-2\cos A>0$ for all $x\in(x_0,((1/2)-\varepsilon)k]$ and all sufficiently large k. From this and (A.11), we conclude that $|T_k(w)|>1$.

Similar arguments were carried out separately for x in the ranges $((\frac{1}{2}-\varepsilon)k,(\frac{1}{2}+\varepsilon)k]$, $((\frac{1}{2}+\varepsilon)k,0.8k]$ and $(0.8k,\infty)$. We omit the computations which are similar to those for $x\in(x_0,(\frac{1}{2}-\varepsilon)k]$.

We have proved the first statement in Lemma 12. For the second, note that, for $w\in C$ with $\Re e(w)>x_0$, we have $T(w)=1-\mathrm{e}^{-\mathrm{i}A}\mathrm{e}^{-x}$, and therefore $T_k(w)-T(w)=T_k(w)-(1-\mathrm{e}^{-\mathrm{i}A}\mathrm{e}^{-x})$. One can verify, in each of the four intervals for x, that $|T_k(w)-(1-\mathrm{e}^{-\mathrm{i}A}\mathrm{e}^{-x})|=\mathrm{o}(1)$.

Hence $T_k\to T$ uniformly for $w\in C$ and $\Re e(w)\geqslant x_0$. □

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