MEM 640 Lecture 3: Zero-Order Hold (ZOH)

Why Discretize?

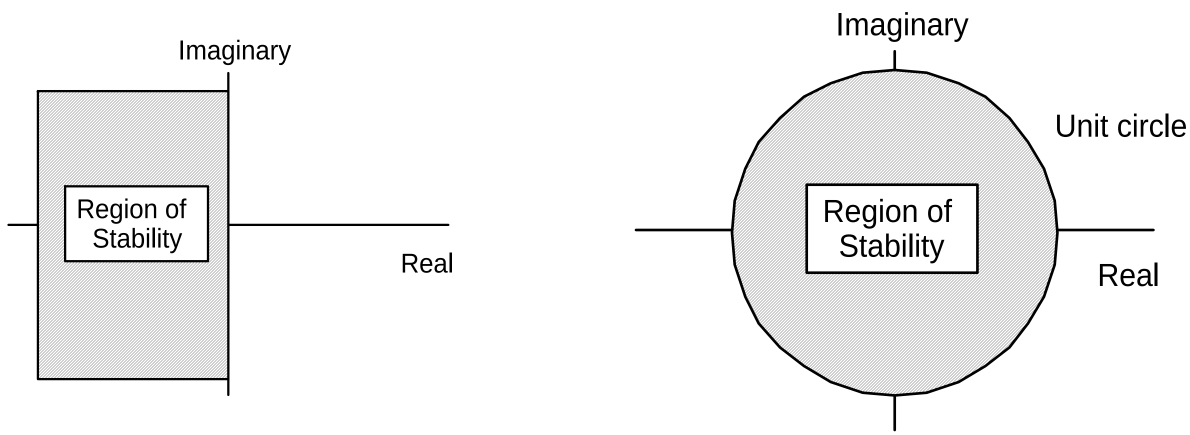
Observation:

- Why is there a distinction between continuous-time and discrete-time?
- Stability: Left-hand s-plane versus inside unit circle
- Digital control performance is dependent on sampling time
- MEM 639 (and MEM 351): implemented computer control, but no z-Transforms?

Intuition:

- If sampling time is zero, discrete-time becomes continuous-time?
- Intuition fails because of the way we discretize

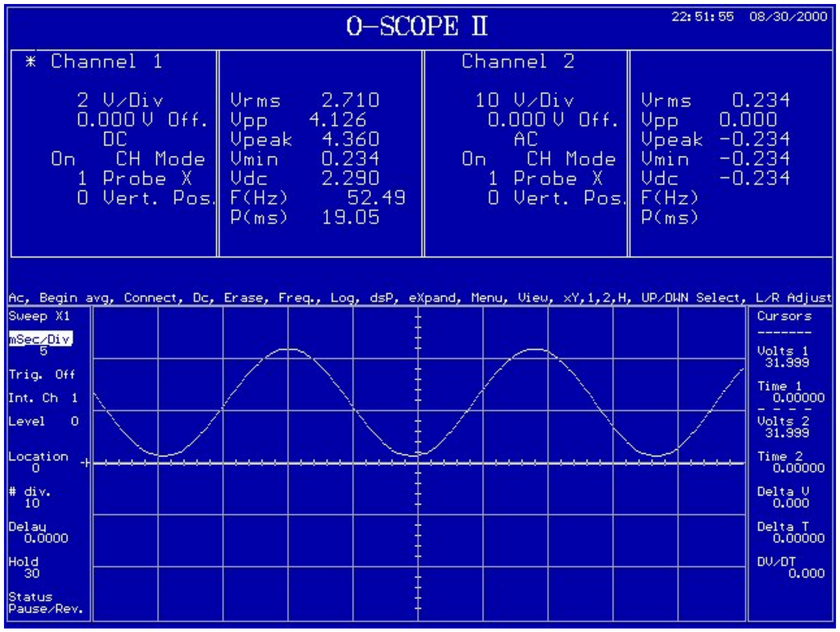
Traditionally $\frac{de(t)}{dt} \approx \frac{e(kT) - e((k-1)T)}{T}$ and $\int_{kT}^{(k+1)T} e(t)dt \approx e(kT) \times T$



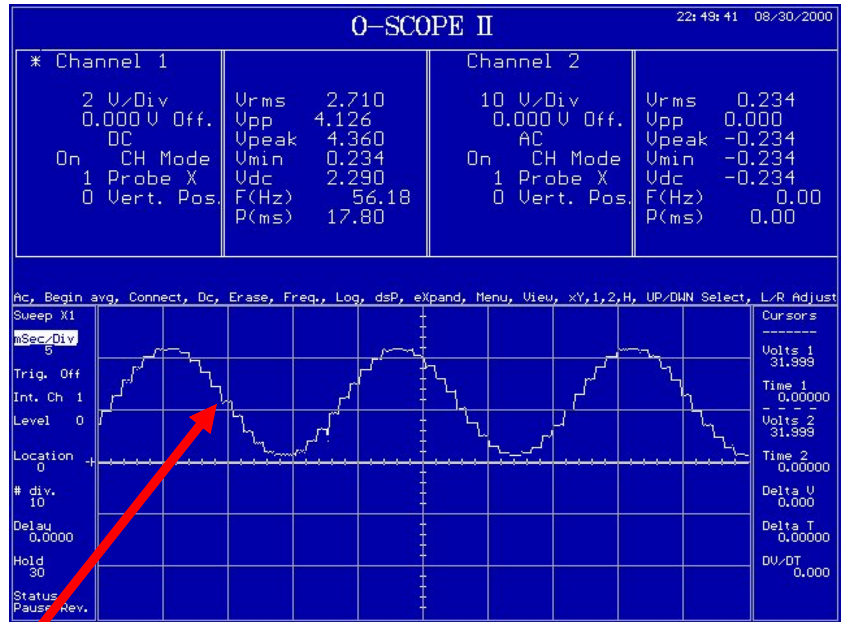
Continuous-Time: s-plane   Discrete-Time: z-plane

What's the Big Deal and Why Care?

Currently: Design with differential equations, Laplace domain, state-space
In other words: Use continuous-time techniques



DAC output looks good



ADC takes time: ZOH Phenomena

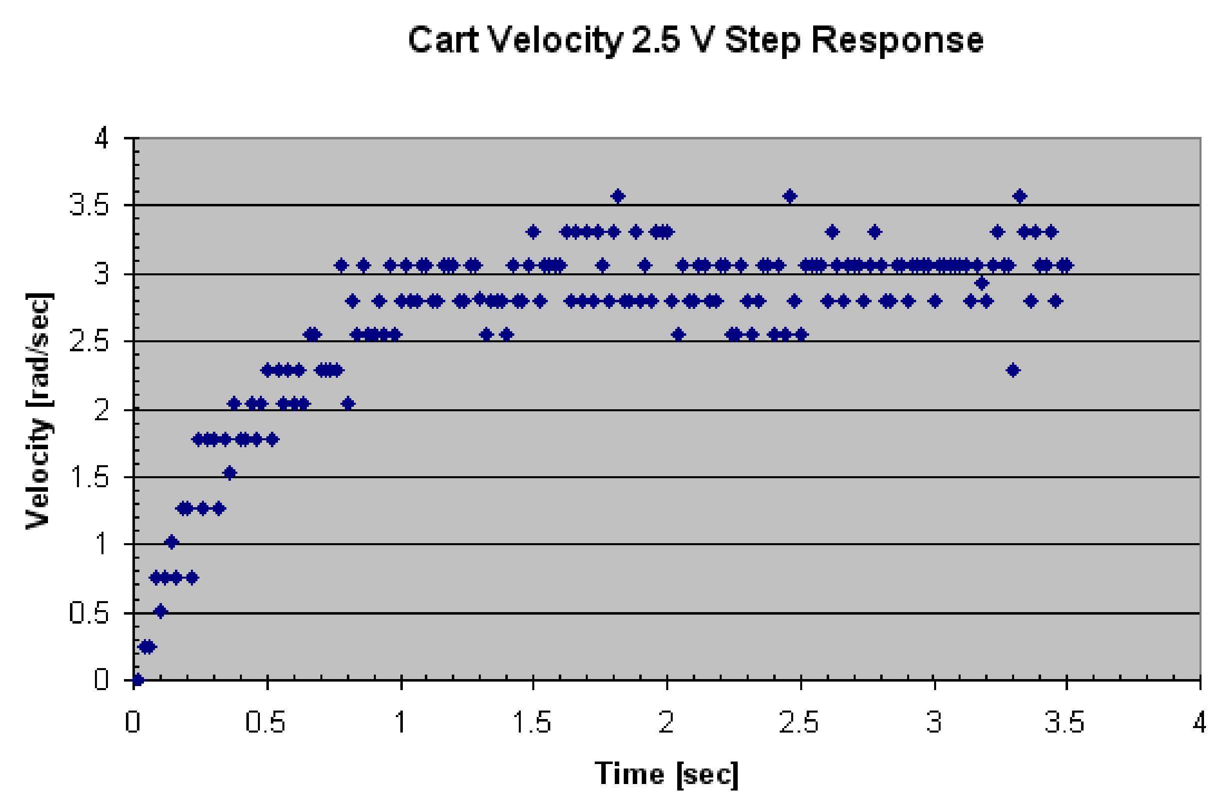
Cannot say what the signal value is in-between sampling times.
This may be troublesome when hold-times are long.

Consequence: Design with difference equations, Z-domain and discrete-time techniques

When Can I Get Away with only CT?

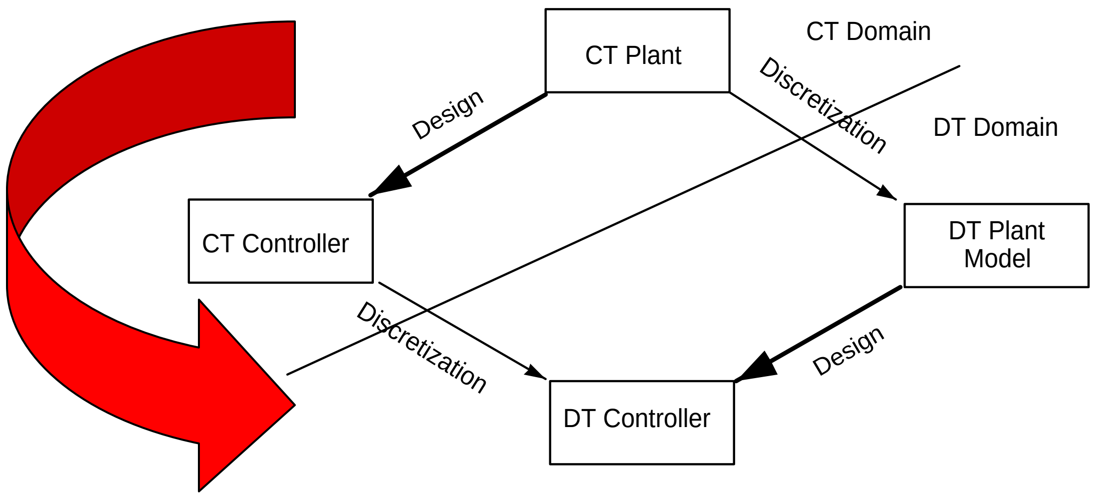
Recall Past Experiences

- Time constant: Step response gives clues on system dynamics
- Rule of thumb: Sampling time should be 4 to 20 times the time constant



Slow system dynamics allows one to get away with continuous-time techniques.

Two Approaches to Digital Design



Approach 1 – Indirect Design

Step 1: design controller in continuous-time (Laplace) domain

Step 2: Discretize to obtain discrete-time controller version

Method: Replace Laplace operator s with an approximate (mapping model)

$$\frac{d}{dt} = \frac{q-1}{T}$$ Forward-difference Model

$$\frac{d}{dt} = \frac{q-1}{Tq}$$ Backward-difference Model

$$\frac{d}{dt} = \frac{q-1}{T}\frac{2}{q+1}$$ Tustin's Model

Pros:

- Can use previously-learned methods Root-locus, Bode, Eigenvalues etc

Indirect Design Example

Problem: Derive the DT version of a PD controller using backward-difference

Solution: Recall, the CT PD controller is given by

$$u(t) = K_P e(t) + K_D \frac{d}{dt} e(t)$$

Differentiation is approximated by the backward-difference as

$$\left.\frac{de(t)}{dt}\right|_{t=kT} \approx \frac{e(k) - e(k-1)}{T}$$

Hence the DT version of PD controller is obtained as

$$u(k) = K_P e(k) + K_D \frac{e(k) - e(k-1)}{T} = \left(K_P + \frac{K_D}{T}\right) e(k) - \frac{K_D}{T} e(k-1)$$

This means that $u(k)$ at $t = kT$ is generated as a sum of the error at the present time (multiplied by a constant) and error at previous time (multiplied by a constant)

Algorithm Implementation in Difference Form

Continuous PID controller given by

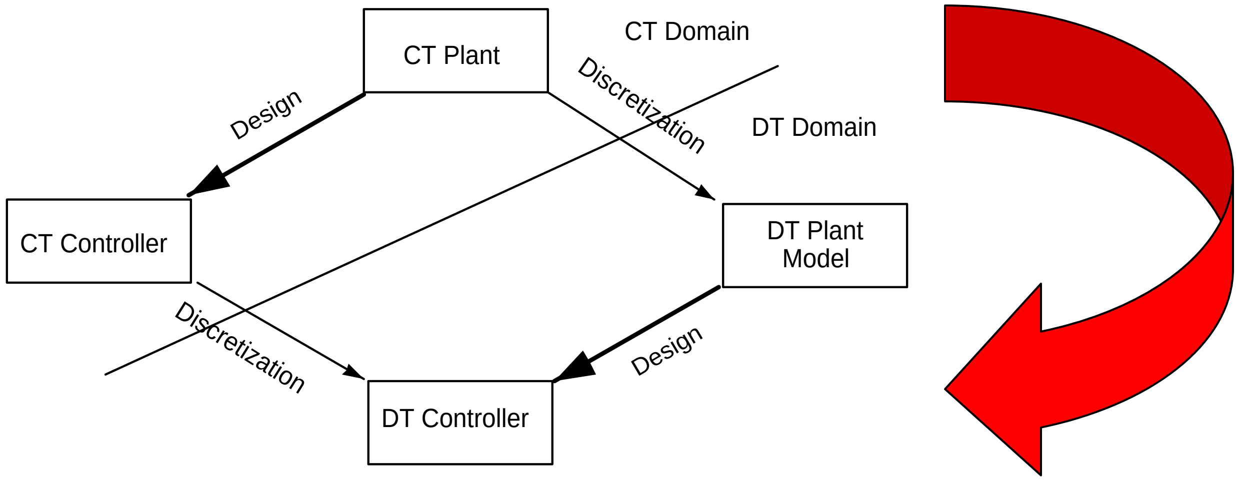
$$u(t) = K_P e(t) + K_I \int e(t)dt + K_D \frac{d}{dt} e(t)$$

```
input Kp;      % proportional gain
input Ki;      % integral gain
input Kd;      % derivative gain
input T;       % sampling interval
I = 0;         % initialization for integral
errTemp = 0;   % initialization
do {
    r = referenceSpeed();
    f = rpm(inportb(adc));    % compute speed from ADC
    err = r – f;              % calculate error
    I = I + err*T;            % integrate error
    D = (err - errTemp)/T;    % differentiate error
    U = Kp*err + Ki*I + Kd*D  % Control signal
    outportb(u);              % send to DAC
    errTemp = err;            % Update errTemp for next interval
} while (time equal to sampling time
```

$$\int_{kT}^{(k+1)T} e(t)dt \approx e(kT) \times T$$

$$\frac{de(t)}{dt} \approx \frac{e(kT) - e((k-1)T)}{T}$$

Two Approaches to Digital Design



Approach 2 – Direct Design

Step 1: Discretize CT Plant using invariant models (Z-transform)

Step 2: Design controller in DT domain

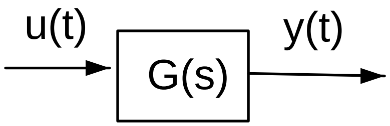
NB: Laplace s-plane becomes unit circle, Routh becomes Jury test etc

$G(z) = Z[G'(s)]$	Impulse invariant model
$G(z) = \frac{z-1}{z} Z\left[\frac{G'(s)}{Ts}\right]$	Step invariant model
$G(z) = \frac{(z-1)^2}{z} Z\left[\frac{G'(s)}{T^2 s^2}\right]$	Ramp invariant model

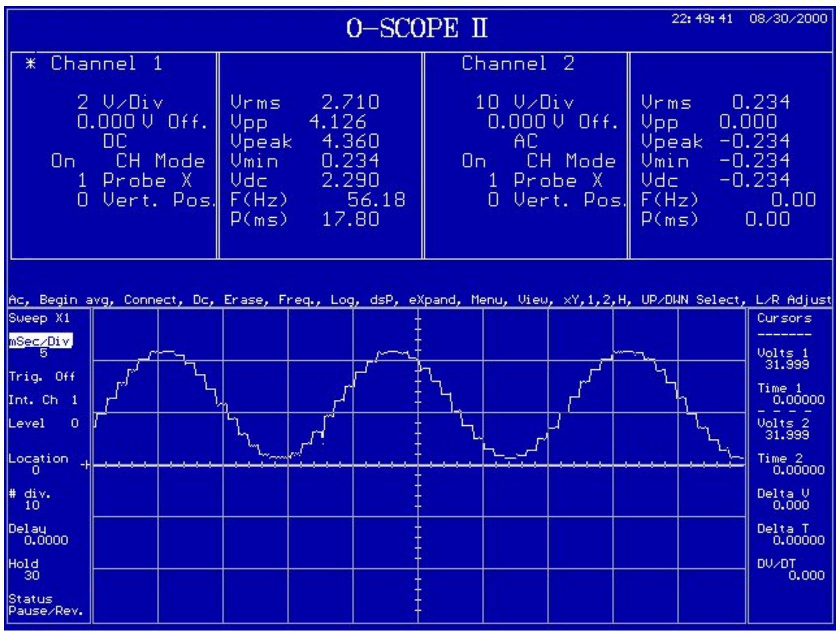
$G'(s)$: CT Transfer Function

$G(z)$: DT Transfer Function

The Need for Z-transforms

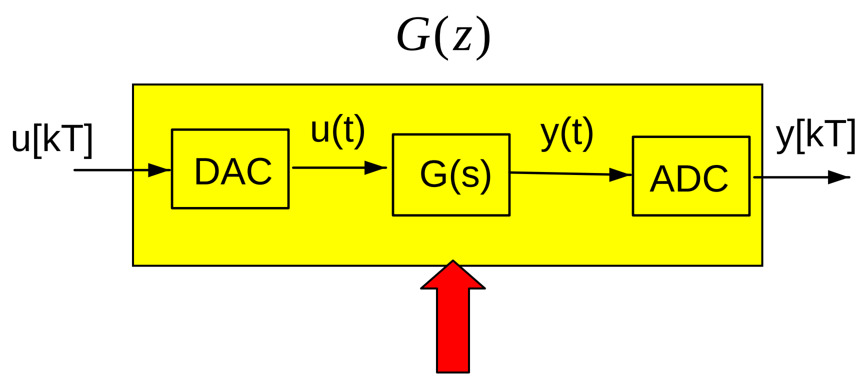


In continuous-time: You design controllers with differential equations (and implement with op-amps), with Laplace transforms, or state-space.



ADC takes time: ZOH Phenomena

In discrete-time: You can design controllers with difference equations (and implement with code), with Z-transforms, or state-space.



Must find discrete version to account for hold phenomena

ZOH Discrete Version: $G(z) = (1 - z^{-1}) Z\left\{ L^{-1}\left[\frac{G(s)}{s} \right] \right\}$ (1)

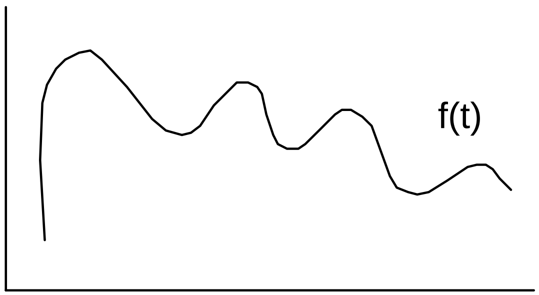
Z-Transforms

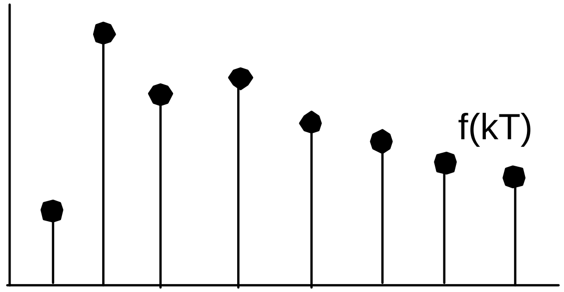
Definition Z-Transform:

$$F(z) = Z\left\{f[kT]\right\} = \sum_{k=0}^{\infty} f[kT]z^{-k}$$

Where $f(kT)$ is the sampled (i.e. discretized) version of $f(t)$

Graphically have:





Mathematically have:

$$f[kT] = \sum_{k=-\infty}^{\infty} f(t)\cdot\delta(t-kT)$$

Here, subject the CT function by an impulse, thus creating sampled version

Problem 1: Calculate the z-transform for the unit step function

Solution: The unit step function given by $f[kT] = 1$

Recall definition:
$$F(z) = Z\{f[kT]\} = \sum_{k=0}^{\infty} f[kT] z^{-k}$$

Hence:
$$F(z) = \sum_{k=0}^{\infty} 1 \cdot z^{-k}$$

But recall that $\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}$ where a is a geometric series

Hence:
$$F(z) = \sum \left(z^{-1}\right)^k = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

Problem 2: Calculate the z-transform for unit pulse

Solution: The unit pulse function given by $\delta[k]$

Recall definition: $$F(z) = Z\{f[kT]\} = \sum_{k=0}^{\infty} f[kT] z^{-k}$$

Hence: $$F(z) = \sum_{k=0}^{\infty} \delta[k] z^{-k}$$

$$F(z) = \delta[0] + \delta[1] z^{-1} + \delta[2] z^{-2} + \cdots = 1$$

Problem 3: Calculate z-transform for unit pulse with shift n to the right (i.e. delay)

Solution: $$F(z) = \sum_{k=0}^{\infty} \delta[k-n] z^{-k}$$

$$F(z) = \delta[n] z^{0} + \delta[1-n] z^{-1} + \delta[2-n] z^{-2} + \cdots + \delta[n-n] z^{-n} + \delta[n+1-n] z^{-n-1} + \cdots$$

$$F(z) = z^{-n}$$

Z-transform Property

If $f[kT+nT]$ then $Z\{f[kT+nT]\}=z^{n}F(z)$

Proof: $f[kT+nT]=f[(k+n)T]$ This is just $f[kT]$ shifted n to the right

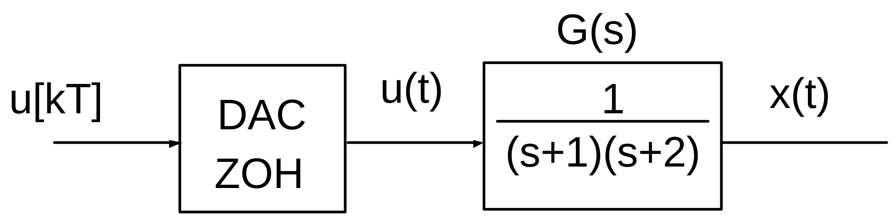
$$f[(k+n)T]=f[kT]\delta[kT+nT]$$

$$Z\{f[(k+n)T]\}=Z\{f[kT]\delta[kT+nT]\}$$

From Problem 3 solution, can say

$$Z\{f[(k+n)T]\}=z^{n}Z\{f[kT]\}=z^{n}F(z)$$

Calculating Difference Equation via Discrete TF G(z)



Problem 4: Find the difference equation relating $u[kT]$ to the output at the sample times $x[kT]$ for any arbitrary unspecified sample time T

Solution: Recall (1) $G(z) = \left(1 - z^{-1}\right) Z\left\{ L^{-1}\left[\frac{G(s)}{s}\right]\right\}$

Step 1: Calculate $L^{-1}\left[\frac{G(s)}{s}\right]$

Use partial fraction expansion $\frac{1}{s}\frac{1}{(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$

Yields $L^{-1}\left\{\frac{1}{2s} - \frac{1}{s+1} - \frac{1}{2(s+2)}\right\} = \frac{1}{2}u(t) - e^{-t} + \frac{1}{2}e^{-2t}$

Step 2: Substitute $t = kT$ and take z-transform

$$Z\left\{L^{-1}\left[\frac{G(s)}{s}\right]\right\} \to Z\left\{\frac{1}{2}u[kT] - e^{-kT} + \frac{1}{2}e^{-2kT}\right\} \quad = \frac{1}{2}\frac{z}{z-1} - \frac{z}{z-e^{-T}} + \frac{1}{2}\frac{z}{z-e^{-2T}}$$

Recall Problems 1 and can use z-transform tables

Step 3: Multiply by $\frac{z-1}{z}$

$$G(z) = \frac{z-1}{z}\left(\frac{z\left(z-e^{-T}\right)\left(z-e^{-2T}\right) - z(z-1)\left(z-e^{-2T}\right)2 + z(z-1)\left(z-e^{-T}\right)}{2(z-1)\left(z-e^{-T}\right)\left(z-e^{-2T}\right)}\right)$$

$$G(z) = \frac{1}{2}\left(\frac{z\left(e^{-2T} - 2e^{-T} + 1\right) + \left(e^{-3T} - 2e^{-2T} + e^{-T}\right)}{z^2 - z\left(e^{-2T} + e^{-T}\right) + e^{-3T}}\right) = \frac{X(z)}{U(z)}$$

Hence have:

$$2\left(z^2 - z\left(e^{-2T} + e^{-T}\right) + e^{-3T}\right)X(z) = \left(z\left(e^{-2T} - 2e^{-T} + 1\right) + \left(e^{-3T} - 2e^{-2T} + e^{-T}\right)\right)U(z)$$

Step 4: Recall that $z^n F(z)$ represents shift in time $f[(k+n)T]$

$$2x[(k+2)T] - 2\left(e^{-2T} + e^{-T}\right)x[(k+1)T] + 2e^{-3T}x[kT] =$$
$$\left(e^{-2T} - 2e^{-T} + 1\right)u[(k+1)T] + \left(e^{-3T} - 2e^{-2T} + e^{-T}\right)u[kT]$$