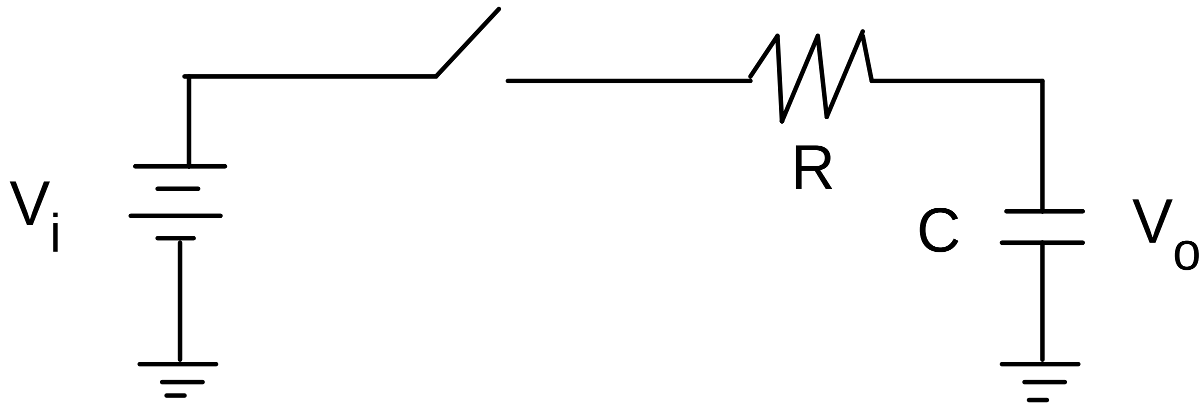
MEM 640 Lecture 1: Filters and Time Response

Filters: Why Study Them?

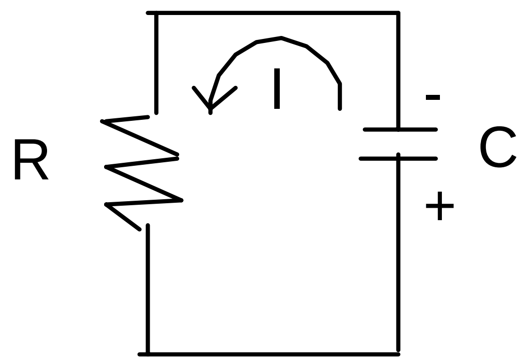


- Low Pass Filter is a classic first-order system
- Easily assembled for experimentation
- Differential equations are easy to derive: time domain
- Frequency domain easy to derive
- Show duality: time and frequency domain

Background Math

Case Study: Calculate the voltage drop over a capacitor

Capacitor: sole purpose is to store electrons. Mechanical analog: reservoir tank



Recall $Q = CV$

and $$I = \frac{dQ}{dt} = C\frac{dV}{dt} + V\frac{dC}{dt} \quad \left(\frac{dC}{dt} \to 0\right) \qquad (1)$$

Also, the cap is discharging across the resistor (hence negative):

$$I = -\frac{V}{R} \equiv C\frac{dV}{dt} \qquad (2)$$

Equation (2) Is a differential equation and hence

$$\frac{dV}{dt} = -\frac{1}{RC}dt \qquad (3)$$

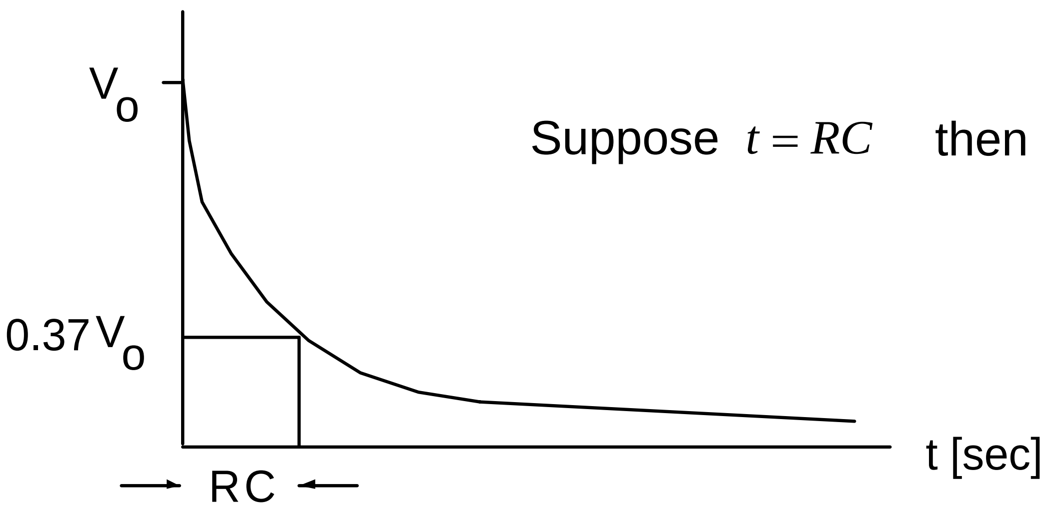
Integrating Equation (3) yields:

$$\ln V = -\frac{1}{RC}t + K_1$$ Where K_1 is a constant

Or

$$V(t) = e^{-t/RC} \cdot e^{K_1} = V_o e^{-t/RC}$$ Where V_o is a constant (4)

Plot of Equation (3) looks like:

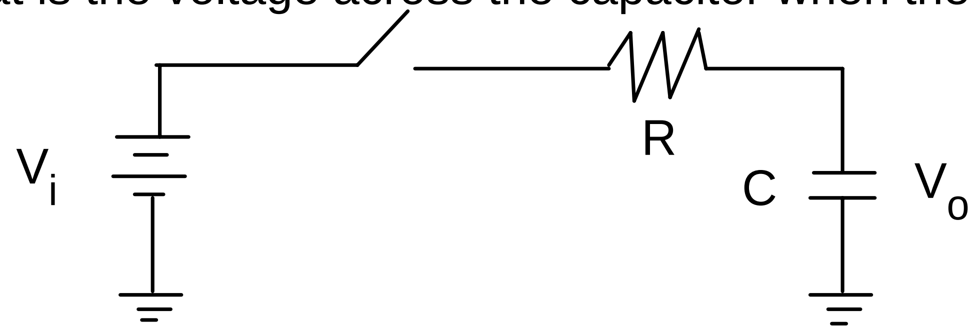


Suppose $t = RC$ then (3) yields: $V(RC) = V_o e^{-1} = 0.37V_o$

At $V(3RC) = V_o e^{-3} = 0.05V_o$ Engineers call the 95% value 3 time constants

Slightly Different Scenario

Case Study: What is the voltage across the capacitor when the switch is closed?



The equation for this circuit is: $I = C\dfrac{dV_o}{dt} \equiv \dfrac{V_i - V_o}{R}$ where V_i is constant (5)

Then

$$\frac{dV_o}{dt} = \frac{1}{RC}\left(V_i - V_o\right) = \frac{V_i}{RC} - \frac{V_o}{RC} \qquad (6)$$

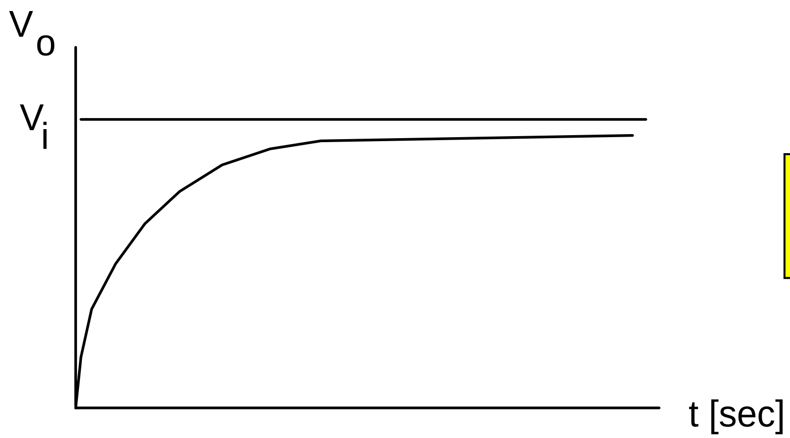
The solution to the non-homogeneous differential equation (6) is:

$$V_o(t) = V_i + Ae^{-t/RC} \qquad (7)$$

Given that $V_o(0) = 0$ then $A = -V_i$ and the solution to (7) becomes

$$V_o(t) = V_i(1 - e^{-t/RC}) \qquad (8)$$

A plot of Equation (8) is



$$V_o(t) = V_i(1 - e^{-t/RC})$$

Note:

$$V(RC) = V_i(1 - e^{-1}) = 0.63V_i$$

and

$$V(5RC) = V_i(1 - e^{-5}) = 0.99V_i$$

A first order system reaches 99% of steady-state in 5 time constants. Also note that the plot of Equation (8) looks like ramp (in early part). That is why a low-pass filter is also known as an <u>integrator</u>

Laplace Domain: Insights into Frequency Response

From (8) we derived the time response for a low pass filter as:

$$V_o(t) = V_i(1 - e^{-at}) \quad \text{where say} \quad a = \frac{1}{RC} \tag{9}$$

Taking Laplace transform of (9) yields

$$V_o(s) = \frac{V_i}{s} - \frac{V_i}{s+a} = \frac{aV_i}{s(s+a)} \tag{10}$$

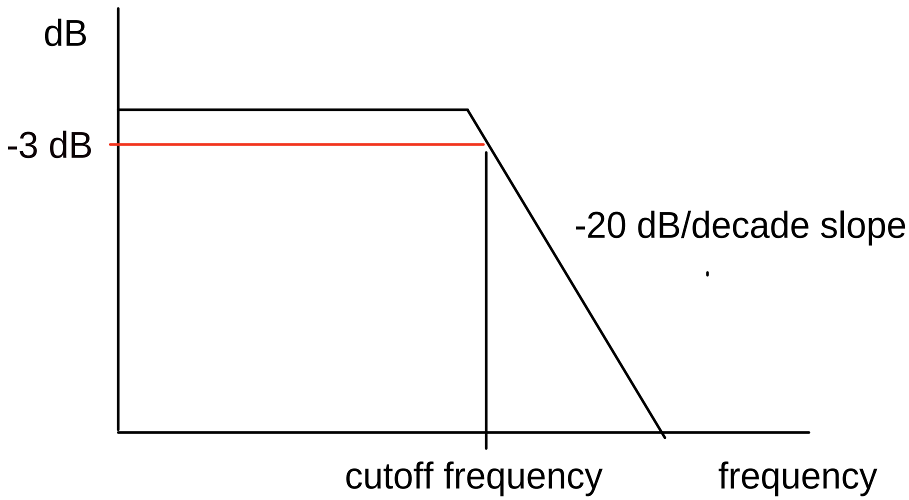
Now since V_i Is a step input, recognize that

$$V_o(s) = \underbrace{\frac{a}{s+a}}_{\text{Actual transfer function}} \cdot \underbrace{\frac{V_i}{s}}_{\text{Step input contribution}}$$

Thus transfer function for a low pass filter is given as

$$\boxed{\frac{V_o}{V_i} = \frac{a}{s+a}} \tag{11}$$

We will see later that the Bode Plot for (11) looks like



$$\frac{V_o}{V_i} = \frac{a}{s+a}$$

Recall that $dB = 20\log x$ thus at -3 dB we have $x = 10^{-3/20} = 0.707$

Also recall from (8) that one time constant yields 63.3% of steady state

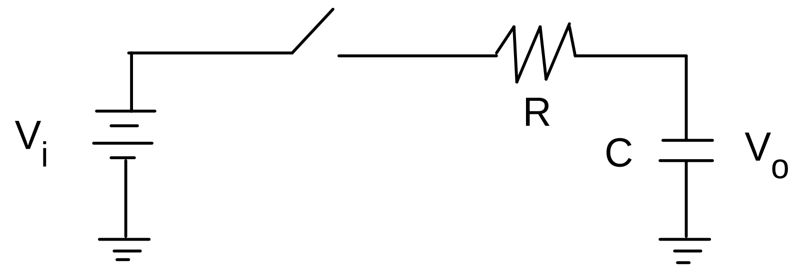
Thus the cutoff frequency is approximately the system's time constant value

Also note that $(0.707)^2 = 0.499$

Power is proportional to voltage squared. Hence the -3 dB point represents a 50% drop in power

Where are we going with this?

Problem: Show mathematically, the integrative properties of the LP filter



Solution: The voltage across the resistor is $V_i - V_o$ So:

$$I = C\frac{dV_o}{dt} \equiv \frac{V_i - V_o}{R} \tag{12}$$

Now suppose that $V_o << V_i$ then (12) can be re-expressed as

$$C\frac{dV_o}{dt} \approx \frac{V_{in}}{R} \tag{13}$$

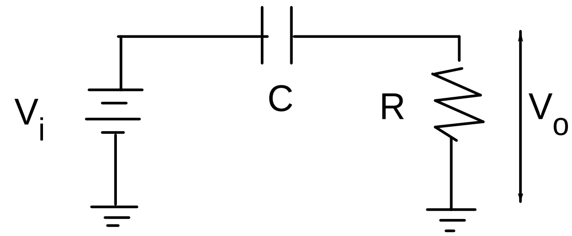
Or,

$$V_o(t) = \underbrace{\int \frac{V_{in}}{RC} dt}_{\text{Integration!}} = \frac{1}{RC}\int^t V_i(t)dt + \text{constant} \tag{14}$$

The approximation (13) says the current is proportional to V_i . If V_i is large and R is large, then we have a current source.

Hi Pass Filters Perform Derivatives

Problem: Show that a hi pass filter perform differentiation



Solution: The voltage across the cap is $V_i - V_o$ so we have

$$I = C\frac{dV}{dt} = C\frac{d}{dt}\left(V_i - V_o\right) \equiv \frac{V_o}{R} \qquad (15)$$

Now if R and C chosen small enough so that

$$\frac{dV_i}{dt} >> \frac{dV_o}{dt} \qquad (16)$$

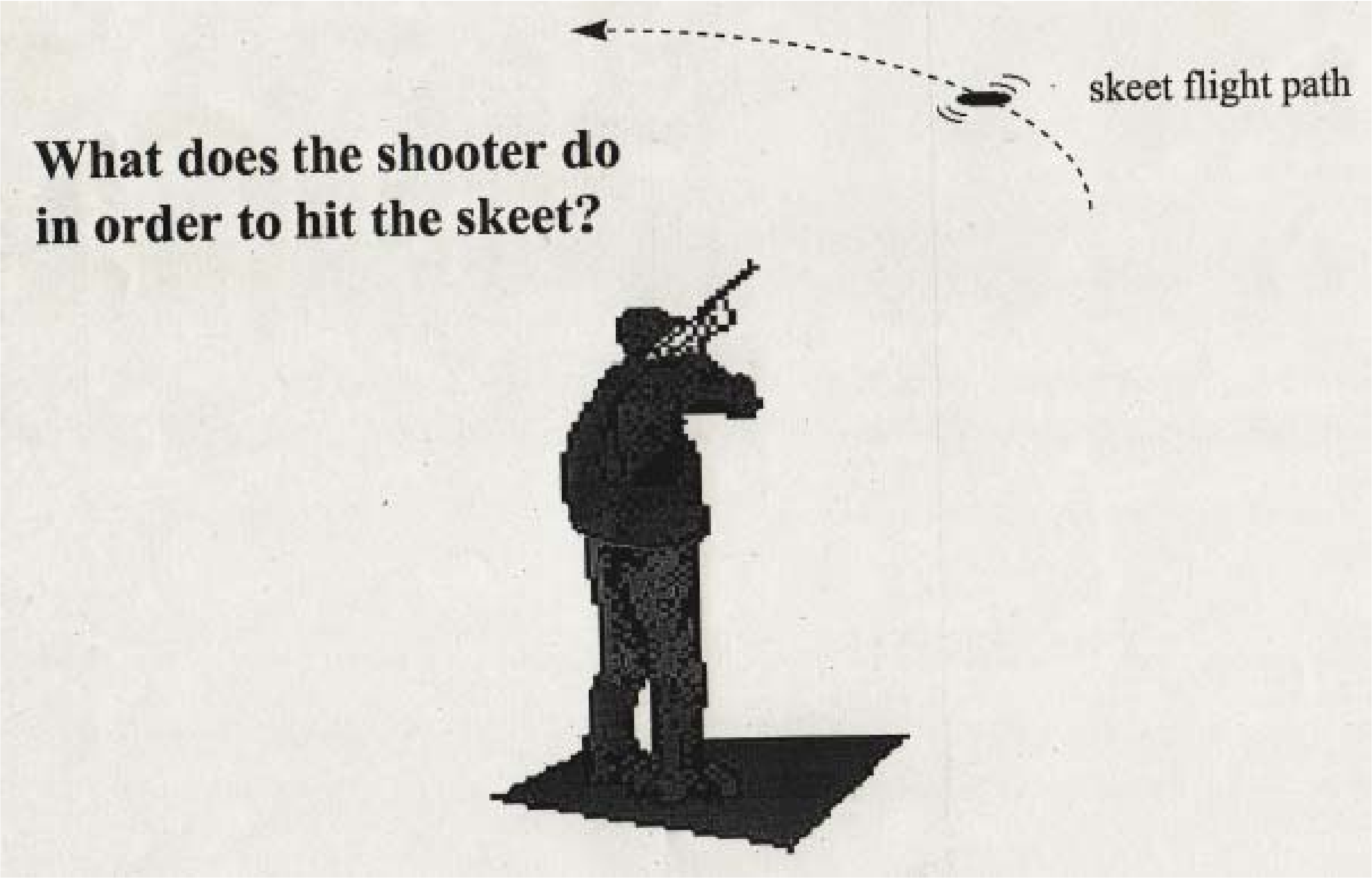
Then

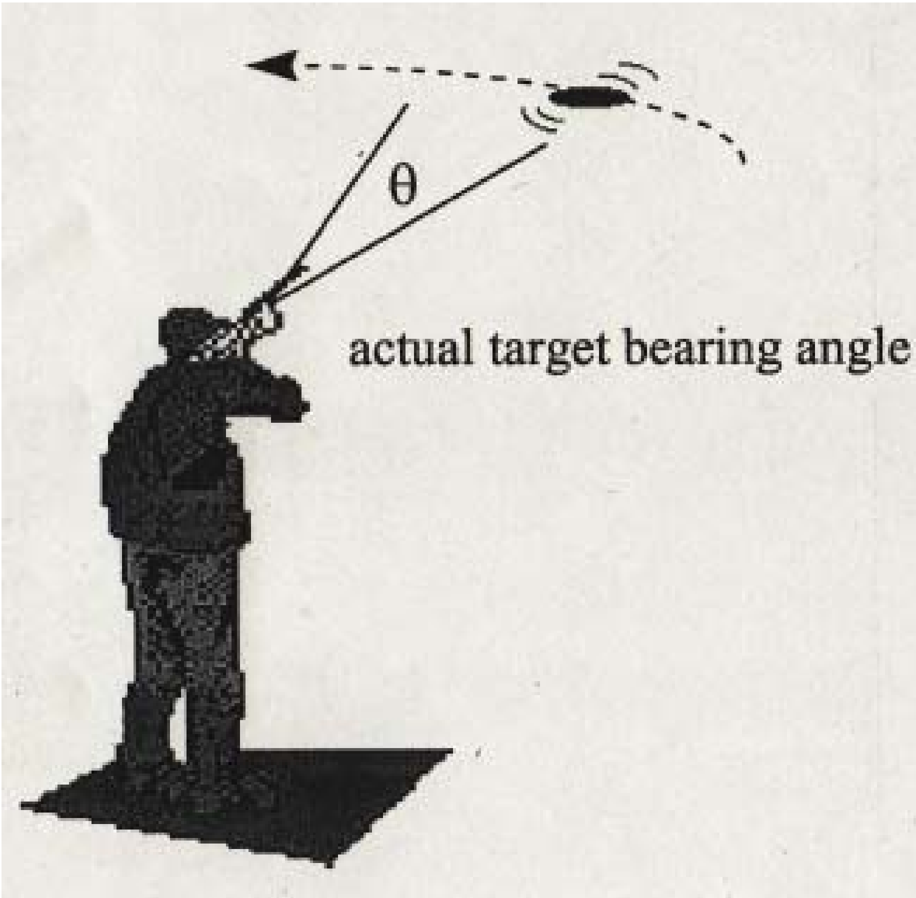
$$C\frac{dV_i}{dt} \approx \frac{V_o}{R} \qquad (17)$$

Hence

$$V_o = \underbrace{RC\frac{dV_i}{dt}}_{\text{Derivative!}} \qquad (18)$$

Grandmother Explanation





Based on Target Dynamics

- Shooter adds angle theta
- Adds phase
- Shooter angle LEADS target angle
- Compensation
- Derivatives: rate of change
- Ignore high frequencies (hi-pass)

Differentiation ≈ LEAD compensation ≈ Hi-Pass Filter