Stability Analysis: Routh Criterion and Root Locus

MEM 639 Real-Time Microcomputer Control 1
Feedback is Good

Recall:

\[ G_{ol}(s) = \frac{b}{s + a} \]

Given step input \( u(t) = 1 \leftrightarrow U(s) = \frac{1}{s} \)

Then

\[ Y(s) = \frac{b}{s + a} \cdot \frac{1}{s} = \frac{b}{s(s + a)} \]

Inverse Laplace yields

\[ y(t) = \frac{b}{a} \left(1 - e^{-at}\right) \]

So output converges

\[ \lim_{t \to \infty} y(t) = \frac{b}{a} \]

Since \( y(t) \neq 1 \) there’s finite error

Compare with (unity) feedback

\[ Y(s) = \frac{G_{ol}}{1 + KG_{ol}} U(s) \]

Step input:

\[ Y(s) = \frac{b}{s(s + a + Kb)} \]

Inverse Laplace:

\[ y(t) = \frac{b}{a + Kb} \left(1 - e^{-(a+Kb)t}\right) \]

If \( K=1 \), output converges

\[ \lim_{t \to \infty} y(t) = \frac{b}{a + Kb} = \frac{b}{a + b} \]

Still finite error – but smaller

So feedback is good!

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Root Locus: A Graphical Tool

Description: Illustrates how the closed-loop pole positions vary as the gain changes from 1 to infinity.

Motivating Example: Proportional Control – Easiest Type of Feedback

Given that \( G_{ol}(s) = \frac{b}{s+a} \) then \( Y(s) = \frac{K_p b}{s(s + a + K_p b)} \)

Root locus plots the CL pole positions (on s-plane) as gain changes.
More Illustrative Example: Pre-cursor to Root Locus

Suppose motor given by:

\[ G_{ol} = \frac{5200}{(s + 10)(s + 1000)} \]

Previous slide says:

\[ Y(s) = \frac{K_p G_{ol}}{1 + K_p G_{ol}} U(s) \]

Consequently CLTF:

\[ G_{cl}(s) = \frac{Y(s)}{U(s)} = \frac{K_p 5200}{(s + 10)(s + 1000) + 5200 K_p} \]

For \( K_p = 2 \)

\[ G_{cl}(s) = \frac{10400}{s^2 + 110s + 11400} \]

For \( K_p = 5 \)

\[ G_{cl}(s) = \frac{26000}{s^2 + 110s + 27000} \]

Try Matlab Program: mtr_p.m

Calculate System properties: \( K_p = 5 \)

\[ \ln \frac{0.3}{0.037} = \zeta \cdot 6.28 \]

\[ \zeta = 0.32 \]

Also, if T=0.0375 sec then

\[ \omega_n = \frac{2\pi}{0.0375\sqrt{1 - 0.32^2}} = 178 \text{ rad/s} \]

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Try Matlab RLTOOLS

For $K_p = 5$

$$G_{cl}(s) = \frac{26000}{s^2 + 110s + 27000}$$

Closed-loop poles:

$$s = -56.2 \pm 154j$$

See report:

$$\zeta = 0.335$$

$$\omega_n = 164 \text{ rad/s}$$

Matches calculations shown on previous slide

So, how does one generate the root locus plot?
Root-Locus Step-by-Step

Problem: Plot the root-locus for the OLTF \( G(s) = \frac{10(Ks + 1)}{s^2 + 8s + 10} \)

Step 1: Recall that \( 1 + KG_o = 0 \) where \( G_o \) is the OLTF

Hence \( 1 + KG_o = 0 \) yields \( 1 + \frac{10(Ks + 1)}{s^2 + 8s + 10} = 0 \) or \( s^2 + 8s + 10 + 10Ks + 10 = 0 \)

Consequently \( K = \frac{-20 - s^2 - 8s}{10s} \) or \( \frac{K10s}{20 + s^2 + 8s} = -1 \)

Therefore \( 1 + KG_o = 0 \) suggests \( 1 + K \left( \frac{10s}{s^2 + 8s + 20} \right) = 0 \)

This is the TF one must work with

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Find poles and zeros of \( G_o(s) = \frac{10s}{s^2 + 8s + 20} \)

Yields poles \( s = -4 \pm 2j \) and zeros \( s = 0 \) \( (1) \)

**Step 2:** Calculate asymptotes where \( \angle s = \pm \frac{i\pi}{n-m} \) and \( i = 1, 3, 5,… \)

\[ G_o(s) = \frac{10s}{s^2 + 8s + 20} \]

\( m = 1 \)

\( n = 2 \)

Have \( n-m \) asymptotes

Hence \( \angle s = \pm i\pi \)

Thus 1 asymptote at: \( s = \pi \) \( (2) \)

**Step 3:** Calculate centroid \( \sigma_c \) where \( \sigma_c = \frac{\sum \text{poles} - \sum \text{zeros}}{n-m} \)

\[ \sigma_c = \frac{(-4 + 2j - 4 - 2j) - 0}{2 - 1} = -8 \] \( (3) \)
As gain changes, the locus due to poles will collide
and asymptotically rides along $\pm \pi$
NB: locus always goes from Pole to Zero

**Step 4:** Calculate breakaway point with $\frac{dK}{ds} = 0$

Recall $K = \frac{-20 - s^2 - 8s}{10s}$  
Hence $\frac{dK}{ds} = \frac{-2s^2 - 8s + 20 + s^2 + 8s}{10s^2}$

And with $\frac{dK}{ds} = 0$ then $s = \pm 2\sqrt{5} = \pm 4.47$  \[(4)\]
Step 5: Calculate angle of departure $\angle KG(s) = 180^0 + K360^0$

where $\angle KG(s) = \sum \angle \text{zeros} - \sum \angle \text{poles}$

$\angle KG(s) = 153.4 - 90 - \theta = 180 + K360$

Hence $\theta = -116.6^0$ (5)

Step 6: Pre-sketch

Root locus will never cross Imaginary axis, so we’re finished

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Routh Stability Criterion

Description: Mathematical trick to assess if a system is asymptotically stable \textit{without} explicitly calculating roots

Motivating Example: Design a PID controller for the following

\[ G_{ol}(s) = \frac{1}{s^5 + 4s^4 + 8s^3 + 9s^2 + 6s + 2} \]

Suppose some poles are unstable. PID-type controllers can’t solve stability. Routh Stability Criterion (RSC) used to determine the number of unstable poles. So one does not waste design time.

**Methodology:**

- If • any coefficient of polynomial is 0
- or • any sign changes in the polynomial coefficients

Then there are unstable or marginally stable roots

Otherwise need to form Routh matrix…
### Constructing the Routh Matrix

Suppose have nth order polynomial \( f(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 \)

Create Routh Matrix:

\[
\begin{array}{cccc|c}
R^n & a_n & a_{n-2} & a_{n-4} & \cdots & 0 \\
R^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots & 0 \\
R^{n-2} & b_1 & b_2 & b_3 & \cdots & 0 \\
R^{n-3} & c_1 & c_2 & c_3 & \cdots & 0 \\
\vdots & & & & & \\
R^2 & d_1 & d_2 & 0 & \cdots & 0 \\
R^1 & e_1 & 0 & 0 & \cdots & 0 \\
R^0 & f_1 & 0 & 0 & \cdots & 0 \\
\end{array}
\]

Where

\[
\begin{align*}
b_1 &= \frac{a_n a_{n-3} - a_{n-1} a_{n-2}}{a_{n-1}} \\
b_2 &= \frac{a_n a_{n-5} - a_{n-1} a_{n-4}}{a_{n-1}} \\
b_3 &= \frac{a_n a_{n-7} - a_{n-1} a_{n-6}}{a_{n-1}} \\
c_1 &= \frac{b_1 b_3 - a_{n-1} b_2}{b_1} \\
c_2 &= \frac{b_1 b_3 - a_{n-1} b_2}{b_1} \\
c_3 &= \frac{b_1 b_3 - a_{n-1} b_2}{b_1}
\end{align*}
\]

Repeat until all remaining \( b_i \) are zero

Also

\[
\begin{align*}
c_1 &= \frac{b_1 b_3 - a_{n-1} b_2}{b_1} \\
c_2 &= \frac{b_1 b_3 - a_{n-1} b_2}{b_1} \\
c_3 &= \frac{b_1 b_3 - a_{n-1} b_2}{b_1}
\end{align*}
\]

Repeat until all remaining \( c_i \) are zero

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**Example:** How many unstable roots for \( f(s) = s^5 + 4s^4 + 8s^3 + 9s^2 + 6s + 2 \)

**Step 1:** Any coefficients zero? No.
Any sign changes in coefficients? No.

**Step 2:** Form Routh Matrix

\[
\begin{array}{cccc}
  s^5 & 1 & 8 & 6 \\
  s^4 & 4 & 9 & 2 \\
  s^3 & b_1 & b_2 & 0 \\
  s^2 & c_1 & c_2 & 0 \\
  s^1 & d_1 & 0 & 0 \\
  s^0 & e_1 & 0 & 0 \\
\end{array}
\]

\[
\begin{align*}
  b_1 &= \frac{4 \cdot 8 - 9}{4} = 5.75 \\
  b_2 &= \frac{4 \cdot 6 - 1 \cdot 2}{4} = 5.5 \\
  c_1 &= \frac{9b_1 - 4b_2}{b_1} = \frac{9(5.75) - 4(5.5)}{5.75} = 5.17 \\
  c_2 &= 2 \\
  d_1 &= 5.5 \\
  d_2 &= 5.5 \\
  e_1 &= 2
\end{align*}
\]

<table>
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<tr>
<th>( s^5 )</th>
<th>( s^4 )</th>
<th>( s^3 )</th>
<th>( s^2 )</th>
<th>( s^1 )</th>
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</tbody>
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Zero sign changes in column. Hence 0 unstable poles.