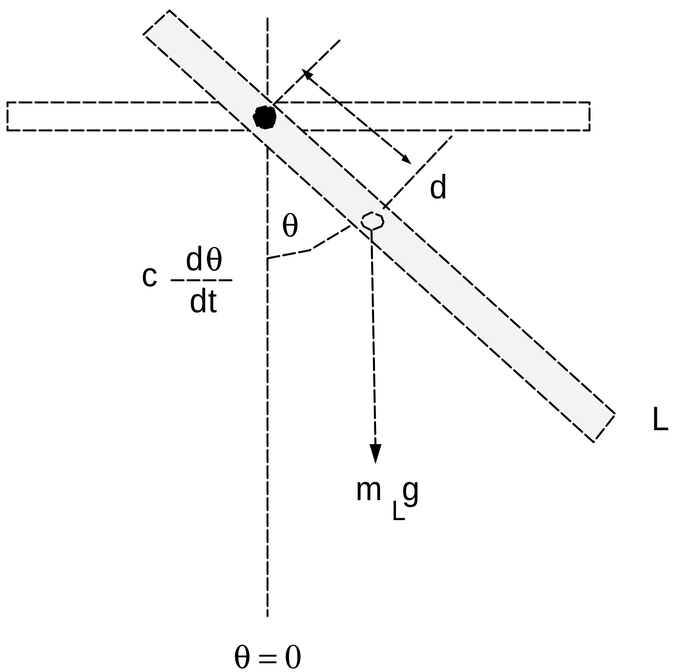
MEM 351 – Dynamic Systems Lab

Representations: Transfer Functions
Poles and Zeros

Recall

Damped Compound Pendulum Equations of Motion



$$\ddot{q} + \frac{c}{J}\dot{q} + \frac{m_L g d}{J} q = 0$$

Linearized 2nd order differential equation assumes **small** angles

L	Bar length [m]
d	Pivot to CG distance [m]
m_L	Mass of pendulum [kg]
J	Moment of Inertia $[kg \cdot m^2]$
c	Viscous damping coefficient $[\frac{Nms}{rad}]$

General 2nd order form

$$\ddot{q} + 2Vw_n\dot{q} + w_n^2 q = 0$$

Tedious Math: Time domain differential equation

2nd order damped system $$\ddot{q} + 2zw_n\dot{q} + w_n^2 q = 0$$

Yields complex roots $$s_{1,2} = -zw_n \pm j \cdot w_n \sqrt{1 - z^2}$$

Time domain solution

$$q_c(t) = e^{-zw_n t}\left\{A_1 \cos\left(w_n t\sqrt{1-z^2}\right) + A_2 \sin\left(w_n t\sqrt{1-z^2}\right)\right\} \qquad (1)$$

Small real root will yield long settling times

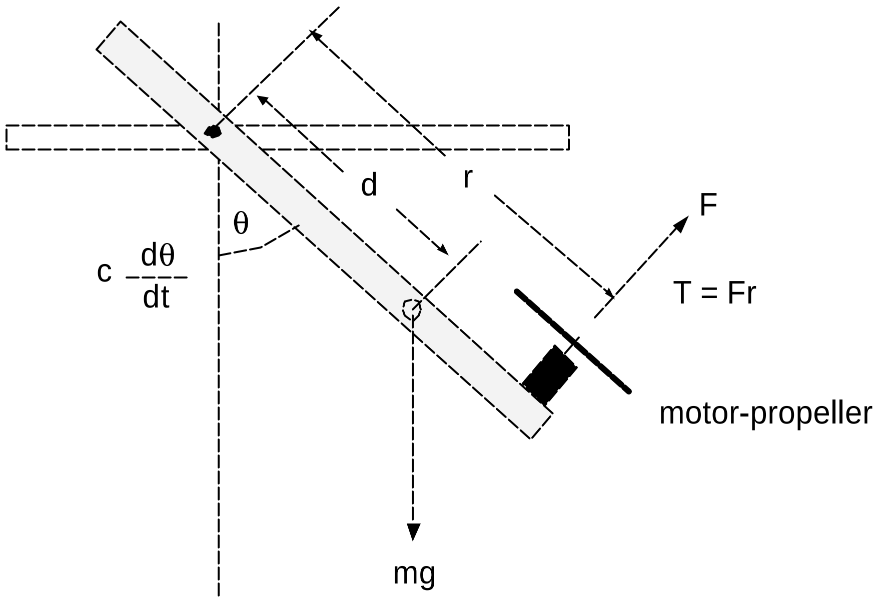
Can be shown:

Time constant $T_c = w_n z$

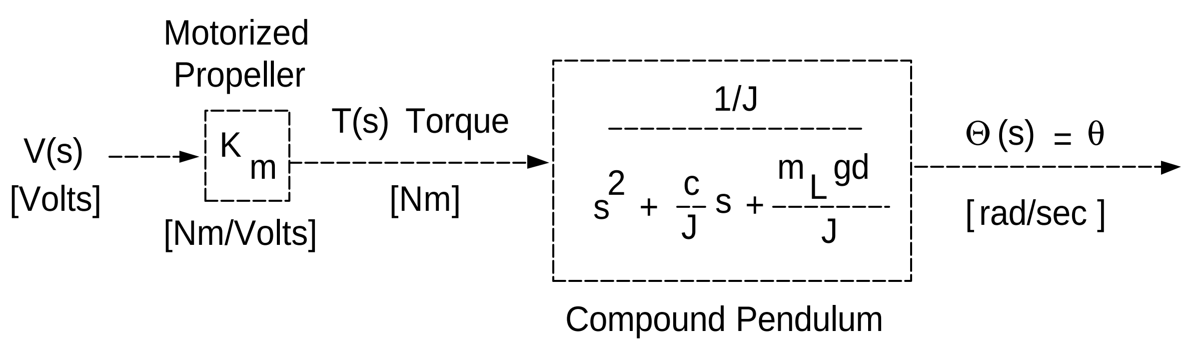
2% settling time $t_s = \frac{4}{T_c}$

(2)

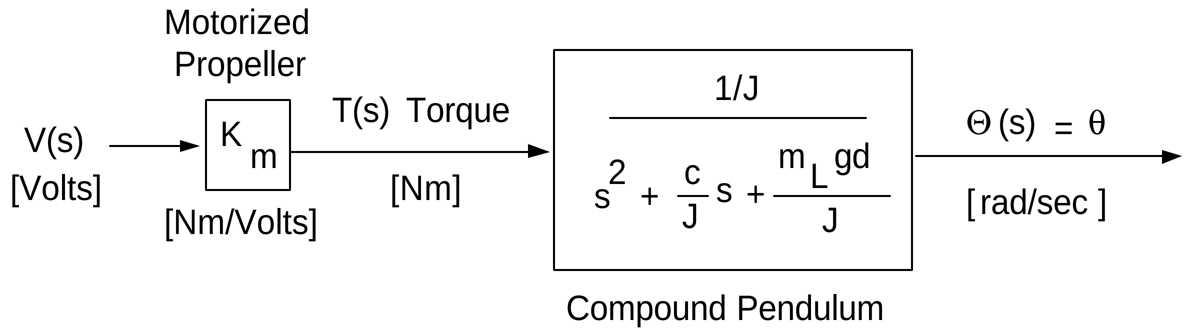
Easier Math I: Laplace Domain



- Voltage $V(s)$ applied to motor
- Propeller spins, creating lift force $F(s)$
- Lift on lever arm *r* creates torque $\mathrm{T}(s)$
- Pendulum rotates angle $\Theta(s)$



Calculating Constants



K_m Theory: can calculate lift force if have propeller pitch and radius dimensions, air density and motor angular velocity.

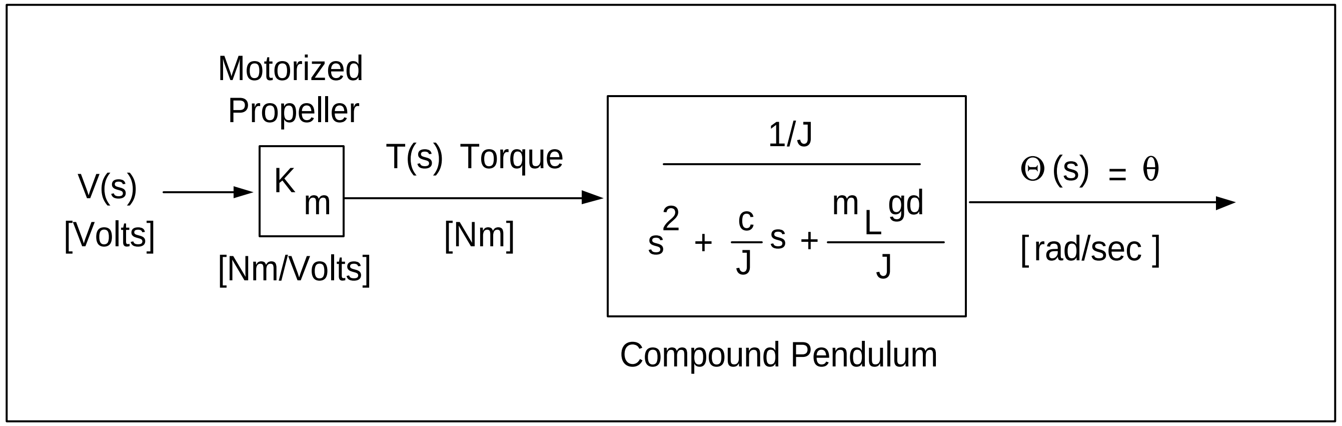
Experimentally, apply known voltage V and pendulum will eventually reach **steady-state**. Recall

$$J\ddot{q} + c\dot{q} + m_L gd \sin q = T$$

At steady-state angular acceleration and velocity are zero. The torque at this known voltage is calculated by:

$$T\big|_{ss} = m_L gd \sin q_{ss}$$ And hence $$K_m = \frac{T\big|_{ss}}{V}$$

Open-Loop Transfer Function



OLTF: $$\frac{\Theta(s)}{V(s)} = \frac{K_m / J}{s^2 + \frac{c}{J}s + \frac{m_L g d}{J}} = G_{ol}(s)$$

Given

$$K_m = 0.017 \quad \text{Nm/V}$$

$$d = 0.023 \quad \text{m}$$

$$J = 0.0090 \quad \text{kgm}^2$$

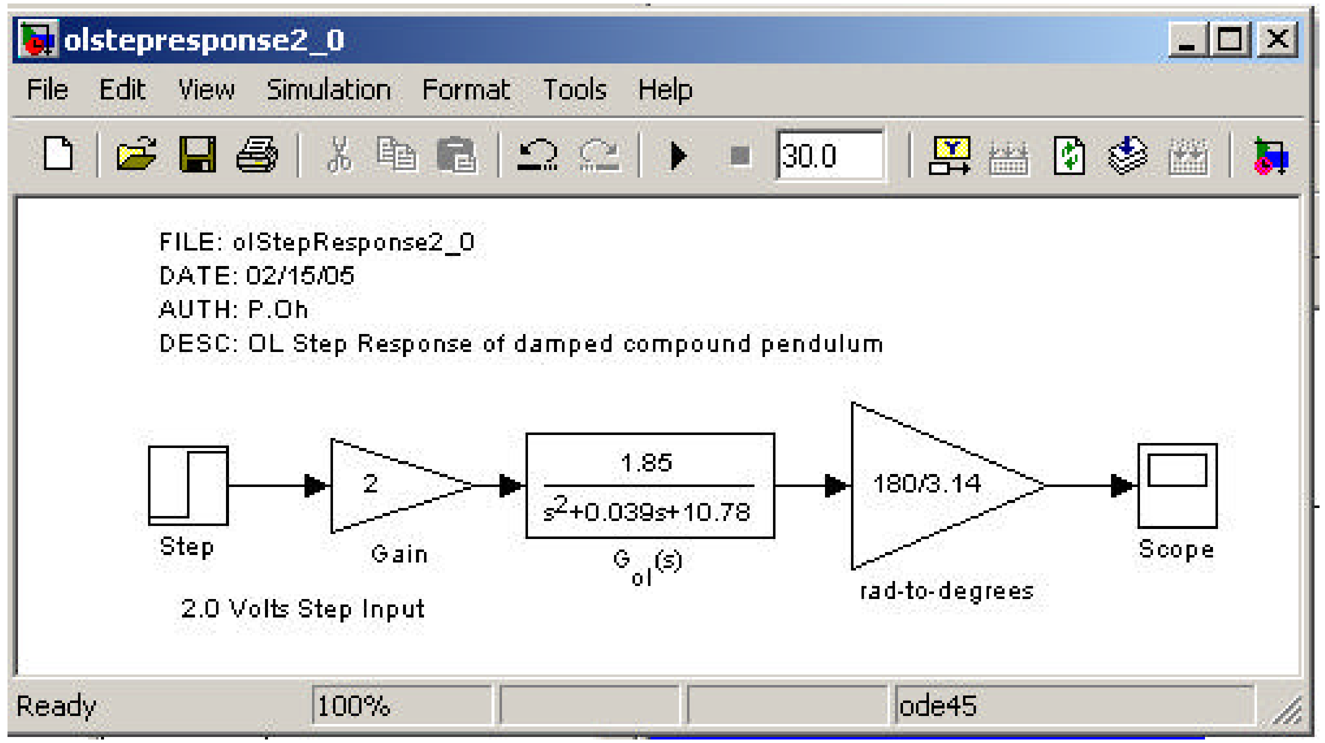
$$m_L = 0.43 \quad \text{kg}$$

$$c = 0.00035 \quad \text{Nms/rad}$$

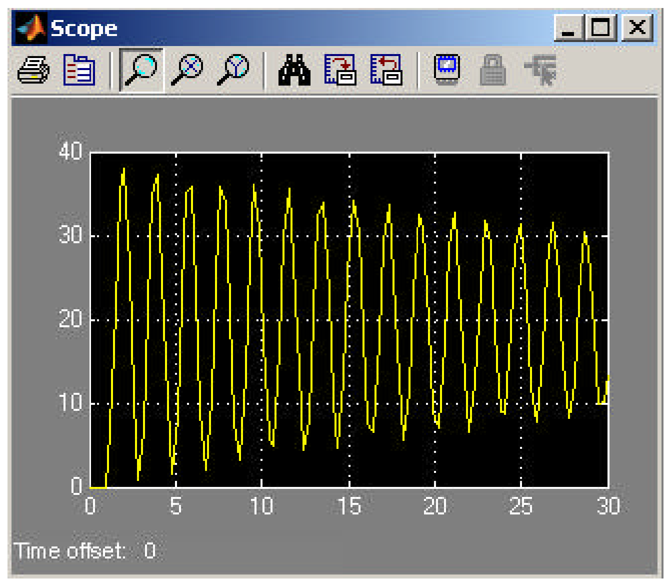
$$\frac{\Theta(s)}{V(s)} = \frac{1.89}{s^2 + 0.039s + 10.77} = G_{ol}(s) \qquad (3)$$

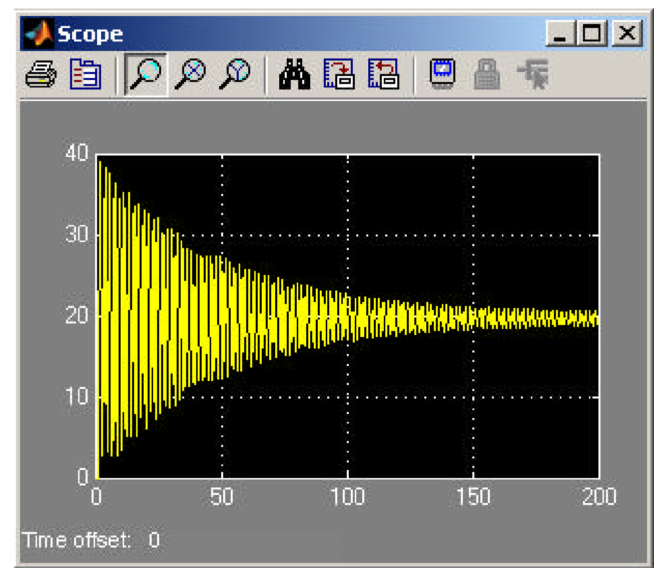
Laplace domain OL Transfer function

OLTF Simulations



Simulink





Simulation reveals long settling time. This is consistent with the low viscous damping coefficient. Poles of the characteristic equation reveal the large oscillations. Recall from (1)

$$\frac{\Theta(s)}{V(s)} = \frac{1.89}{s^2 + 0.039s + 10.77} = G_{ol}(s)$$

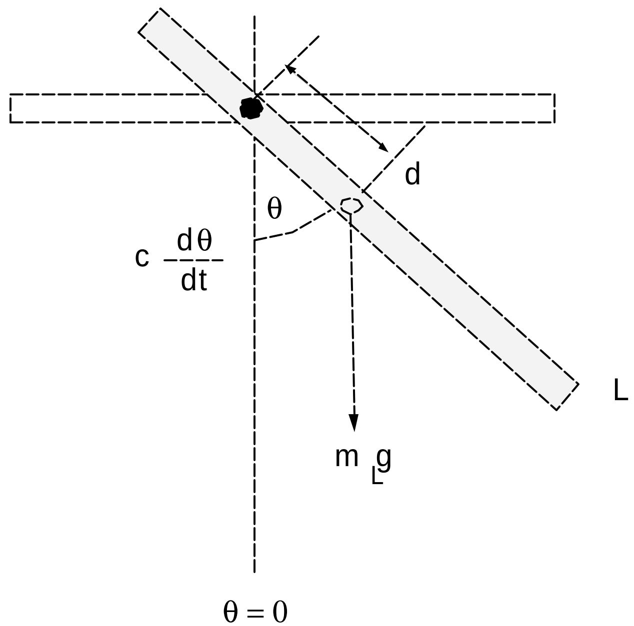
Roots of the denominator (i.e. the poles) are:

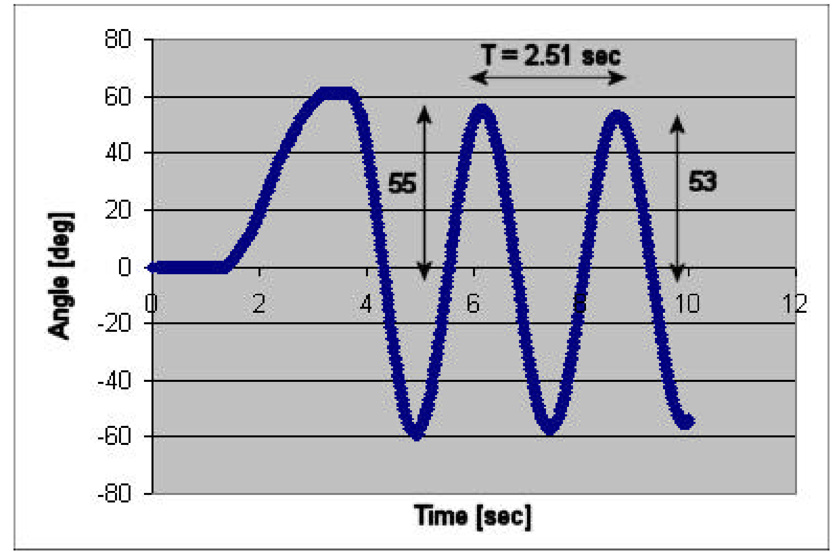
$$s_1 = -0.0019 + j3.28$$
$$s_2 = -0.0019 - j3.28$$

Small real root will yield long settling times

In other words, the system is bordering on the margin of stability. Recall Equations (1) and (2).

System Poles and Zeros – What are they?





$$\ddot{q} + 2Vw_n\dot{q} + w_n^2 q = 0 \qquad (1)$$

$$\ln\frac{a}{b} = \frac{V\,2p}{\sqrt{1-V^2}} = \frac{1}{N}\ln\frac{X_1}{X_{N+1}} \qquad (2A)$$

$$\frac{2p}{T} = w_n\sqrt{1-V^2} \qquad (2B)$$

Control Designer's Goal:

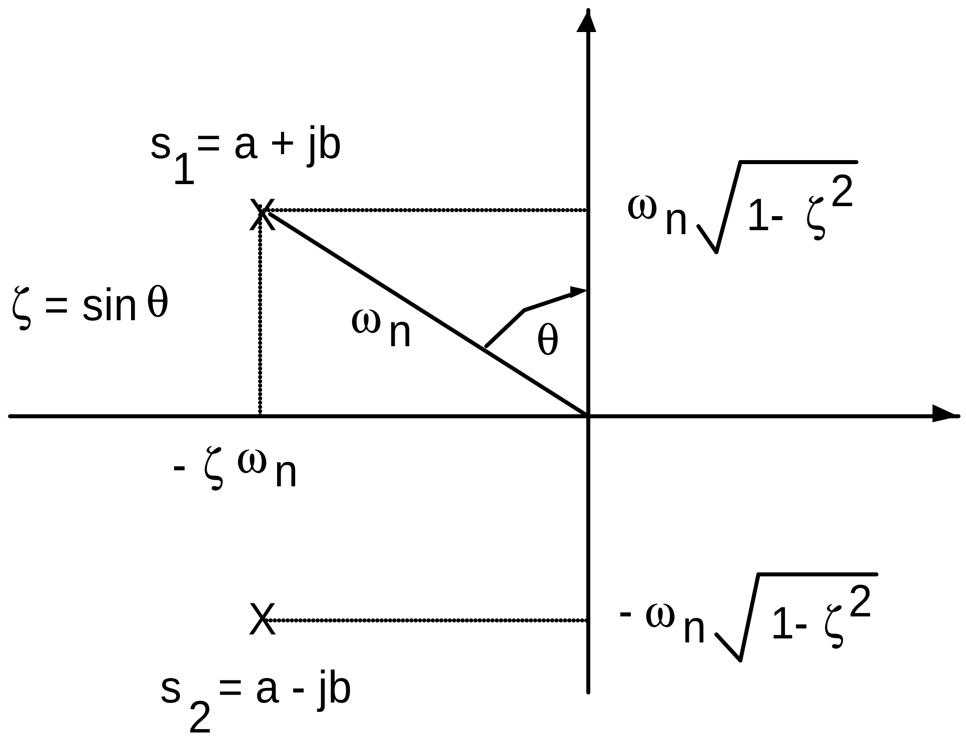
Create compensators that yield desired damping and rise time.

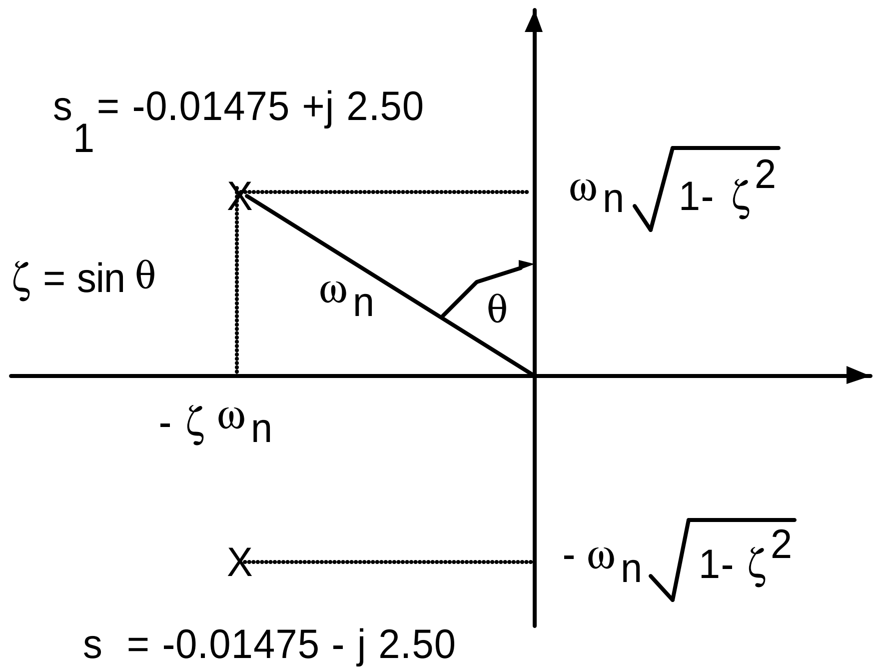
In other words, place poles where one wants them

Calculated the following:

$$w_n = 2.50 \quad \text{rad/s}$$
$$z = 0.0059$$

$$\ddot{q} + 2Vw_n\dot{q} + w_n^2 q = 0$$
$$s^2 + 0.0295s + 6.25 = 0$$
$$s_{1,2} = -0.01475 \pm 2.5j \quad \textbf{poles}$$





$$q = \arctan\frac{0.01475}{2.50} = 0.0059$$

$$z = \sin q = \sin 0.0059 = 0.0059$$

$$w_n = \sqrt{a^2 + b^2} = \sqrt{0.01475^2 + 2.50^2} = 2.50$$

Matches experimental data!

Where are we going with this?

It's called the characteristic equation because it connotes system properties

Poles are the roots of the characteristic equation. As such, the describe stability through w_n and z

Question: can we alter the locations of the poles? If we can, then we change the characteristic of the system…

Answer: This is exactly what the control engineer does. One popular method is called "pole placement" control