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Inventory Models with Shelf-Age and Delay-Dependent Inventory Costs

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In this paper, we show how any model with a general shelf-age-dependent holding cost and delay-dependent backlogging cost structure may be transformed into an equivalent model in which all expected inventory costs are level dependent. We develop our equivalency results, first, for periodic review models with full backlogging of stockouts. These equivalency results permit us to characterize the optimal procurement strategy in various settings and to adopt known algorithms to compute such strategies. For models in which all or part of stockouts are lost, we show that the addition of any shelf-age and delay-dependent cost structure does not complicate the structure of the model beyond what is required under the simplest, i.e., linear, holding and backlogging costs. We proceed to show that our results carry over to continuous review models, with demands generated by compound renewal processes; the continuous review models with shelf-age and delay-dependent carrying and backlogging costs are shown to be equivalent to periodic review models with convex level-dependent inventory cost functions.

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1. Introduction

After initial discussions in seminal textbooks like Hadley and Whitin (1963) and Naddor (1966), inventory theory quickly settled on the paradigm that inventory carrying costs are to be assessed as level-dependent costs, i.e., a (possibly nonlinear) function of the prevailing total inventory level, irrespective of its age composition. Recent papers have pointed out that this representation is inadequate in several important contexts. For example, in the typical settings where all or part of holding costs consist of capital costs, these depend on the items’ purchase price and the prevailing interest rate. When the latter two vary with time, this necessitates decomposing inventory levels according to the times at which the items are purchased, i.e., the shelf ages of various inventory units. This approach was followed in Federgruen and Lee (1990), Levi et al. (2008b), and Stauffer et al. (2011).

In addition, under various financing schemes, the capital costs incurred for a specific order accumulate at interest rates that increase as a function of the amount of time elapsed since the placement or delivery of the order. This problem applies to many trade credit schemes, in particular, where the supplier agrees to being paid when the units are sold to the buyer’s customers. Gupta and Wang (2009) and Federgruen and Wang (2012) modeled such trade credit schemes and discussed their prevalence in today’s economy. Yet another setting where the inventory cost rate for any stocked item increases with the item’s shelf age occurs when the supplier subsidizes part of the inventory costs. For example, in the automobile industry, manufacturers pay the dealer so-called “holdbacks,” i.e., a given amount for each month a car remains in the dealer’s inventory, up to a given time limit (see, e.g., Nagarajan and Rajagopalan 2008). Nahmias et al. (2004) consider settings where the retail price of an item declines as a function of its shelf age at the time of its sale; this price reduction can be modeled as a shelf-age-dependent carrying cost.

To capture the various shelf-age-dependent cost structures, we assume that the marginal inventory cost of a single unit ordered in period $n$ is $\alpha_n(j)$ when reaching a shelf age of $j$ periods; here the functions $\alpha_n(\cdot)$ are general, merely assumed to be nondecreasing.

Backlogging cost rates may also vary with the amount of time by which delivery of a demand unit is delayed. This pattern may result from the structure of contractually agreed upon penalties for late delivery or, in case of implicit backlogging costs, because customers become more impatient over time. See Chen and Zheng (1993), Perry and Stadje (1999), Rosling (2002), and Huh et al. (2011) for examples. To capture these various settings, we assume that the marginal backlogging cost of a unit, demanded in
period \( n \), that has been backlogged for \( j \) periods is given by a quantity \( \beta_n(j) \), with \( \beta_n(\cdot) \) general, nondecreasing functions.

We show how periodic and continuous review models with general shelf-age-dependent holding costs and delay-dependent backlogging costs may be transformed into an equivalent “standard” model in which all expected inventory costs are level dependent. These equivalency results allow for the rapid identification of the structure of an optimal policy, in various models. It also allows for the immediate adoption of algorithms to compute optimal policies.

We first develop our equivalency results for a general model that includes full backlogging, lost sales, or a partial backlogging dynamic as special cases. We proceed to tailor these results to those special cases. Models with full backlogging allow for a one-dimensional state representation via the so-called inventory position, i.e., on-hand inventory plus outstanding orders backlogs. These equivalency results permit us to characterize the optimal procurement strategy in various settings. For example, assuming demands are independent with exogenously given distributions, a simple time-dependent \((s,S)\) policy, acting on the inventory position, is optimal under fixed-plus-linear order costs. When the cost parameters and demand distribution depend on a state-of-the-world variable that evolves according to an independent Markov chain, a state-dependent \((s,S)\) policy is optimal. In the absence of fixed delivery costs, this structure further simplifies to that of a base-stock policy. In the special case where all model parameters are stationary, we show that the base-stock levels increase as we progress to the end of the planning horizon; moreover, these levels can be determined myopically. Other model variants and associated optimal strategies are addressed as well.

We also show that in periodic review models with full backlogging and general shelf-age-dependent inventory costs, the model is equivalent to one with linear holding costs, however, with a specific extended random leadtime distribution.

We then apply our equivalency results to models in which all or part of backlogs is lost. Here, even with the simplest, i.e., linear, holding and backlogging costs, the state of the system needs to be described by an \((L+1)\)-dimensional vector, keeping track of the inventory on hand and all outstanding orders from the last \(L\) periods separately. We show that the addition of any shelf-age and delay-dependent costs does not complicate the structure of the model beyond what is required under the simplest, i.e., linear, holding and backlogging costs. As a result, we generalize various structural optimality results known to prevail under linear inventory costs.

In §4, we show that the above results carry over to continuous review models with general compound renewal demand processes. We show that under minor assumptions, the model is equivalent to a periodic review model with convex inventory level-dependent carrying and backlogging costs. Under fixed-plus-linear costs, this implies, for example, that an \((s,S)\) policy is optimal.

There are two important differences between shelf-age-dependent carrying costs and delay-dependent backlogging costs, in terms of establishing an equivalence with standard level-dependent cost structures. First, for shelf-age-dependent cost structures, the equivalence is achievable without any restrictions, whatsoever. For the delay-dependent backlogging costs, we demonstrate that such an equivalence can only be obtained under an assumption guaranteeing either that no demand unit is delayed by more than the leadtime plus one period or that the incremental backlogging cost rate is constant for delays in excess thereof. The first condition is equivalent to assuming that the inventory position after ordering is always nonnegative.

Second, the equivalence for the backlogging costs is obtained by decomposing the aggregate backlog levels in terms of their age composition; as far as shelf-age-dependent carrying costs are concerned, it is possible to achieve the same equivalence via a parallel decomposition of inventory levels in terms of their shelf ages, but only in periodic review models. However, in the more general continuous review models, we explain that a fundamentally different approach is needed, i.e., one that disaggregates orders in terms of which part of each order reaches any given shelf age.

Section 2 presents a brief literature review. We consider periodic review models with full backlogging, lost sales, or partial backlogging in §3 and study continuous review models with compound renewal demand processes in §4. Section 5 contains concluding remarks. All proofs are deferred to the appendix.

2. Literature Review

Beyond the above referenced applications of shelf-age-dependent carrying costs and delay-dependent backlogging costs, there are three papers that have identified the structure of an optimal policy in special settings. Gupta and Wang (2009) showed that a base stock policy is optimal, under linear order and linear backlogging costs and marginal inventory carrying cost rate functions \(\alpha_n(\cdot)\) that are constant from a certain age on. This result arises as a special case of Theorem 2(a) below. Huh et al. (2011) focused on delay-dependent backlogging costs, assuming standard carrying costs. They established an equivalence with a standard inventory model, in the case of full backlogging, stationary cost parameters and demand distributions, and fixed-plus-linear order costs. (These authors also considered level-dependent backlogging costs, including the possibility of a fixed penalty in any period in which a stockout occurs.) We show that the equivalency results can be achieved in much greater generality. In particular, we develop the equivalency for a model with general partial backlogging options, in which the full backlogging and lost sales cases arise as special cases. After developing the general equivalence result, we show how additional
3. Periodic Review Models

In this section, we consider a general, single item periodic review inventory planning problem with a finite or infinite horizon. As in standard inventory models, we assume that demands in different periods are independent of each other. Orders arrive after a lead time of \( L \) periods. To simplify the exposition, we assume that the lead time \( L \) is deterministic. However, in the online appendix (available as supplemental material at http://dx.doi.org/10.1287/opre.2015.1369) we extend our results to stochastic leadtime processes that are exogenous and sequential. As in standard inventory models, we assume the following sequence of events in each period:

i) at the beginning of the period, all units ordered \( L \) periods ago arrive and remain in stock until sold;

ii) immediately thereafter, the beginning inventory level and all outstanding orders are observed and a new order may be placed;

iii) the period’s demands occur; and

iv) on-hand items are used to clear backlogs and to satisfy demand, and inventory costs are assessed thereafter.

The planning problem is meaningful only if it consists of at least \( L + 1 \) periods. It is therefore convenient to denote the length of the planning horizon by \( N + L \), with \( N \leq \infty \). All costs are discounted with a discount factor \( \rho \leq 1 \). An item ordered in period \( n \) accrues an incremental carrying cost rate \( \alpha_n(j) \) when reaching a shelf age of \( j \) periods, i.e., when still in stock at the end of period \( n + L + j - 1 \). (To allow for settings where the planning horizon starts with a positive inventory, we extend the definition of \( \alpha_n(\cdot) \) for period indices \( n < 0 \).) \( \beta_n(j) \) denotes the incremental backlogging cost rate when a unit demand at period \( n \) waits for \( j \) periods. We assume that at any point in time, the incremental carrying costs of any unit in stock increase with its shelf age and older outstanding demands incur higher incremental backlogging costs; i.e.,

\[
\begin{align*}
\text{(ODY—Old Dearer than Young): } & \quad \alpha_n(j) \geq \alpha_{n+1}(j - 1) \geq \cdots \geq \alpha_{n+j}(0) = 0 \text{ for integer } n \text{ and } j \geq 1. \\
\text{(ODY-B): } & \quad \beta_n(j) \geq \beta_{n+1}(j - 1) \geq \cdots \geq \beta_{n+j}(0) = 0 \text{ for integer } n \text{ and all } j \geq 1.
\end{align*}
\]

These conditions, satisfied in all applications mentioned in §1, ensure that it is optimal to deplete inventories and fill outstanding demands on a first-in-first-out (FIFO) basis. (The online appendix of this paper shows that the equivalency results break down under last-in-first-out depletion of inventories.) Let

\[
\begin{align*}
D_n & = \text{demand in period } n; \\
D[t_1, t_2] & = \text{cumulative demand in periods } t_1, t_1 + 1, \ldots, t_2, \text{ with } t_2 \geq t_1; \\
I_n^b & = \text{the beginning inventory level of period } n \text{ (after inclusion of any order placed one leadtime earlier), i.e., inventory in stock minus backlogs, if any;} \\
l^b_n(D_n, I_n^b) & = \text{the part of period } n \text{'s demand that is backlogged to the next period, if any, when period } n \text{ starts with a beginning inventory } I_n^b; \\
q_n & = \text{order placed in period } n;
\end{align*}
\]

Remarkably, the structure of the optimal policy in perishable item models is very complex, even in the simplest settings with linear costs. This realization, obtained in the 1970s, caused researchers to focus on the identification and evaluation of heuristic policies thereafter. As shown below, in our model with a general shelf-age-dependent cost structure, the existence of an optimal policy of relatively simple structure can be demonstrated in many settings.
The backlogged demand at the end of period $n+L$ that has been delayed for exactly $j$ periods must be part of the set of units initially backlogged in period $n + L - j + 1$, i.e., $I_{n+L-j+1}(D_{n+L-j+1}, l_{n+L-j+1}^b)$. Moreover, $B_{n+L-j}$ represents the part not filled by any orders that arrive during $[n + L - j + 2, n + L]$, which is $\sum_{i=n-j+2}^{n} q_i$. Equation (4) follows because, based on FIFO, the aggregate orders $\sum_{i=n-j+2}^{n} q_i$ are used to fill any backordered demands at the beginning of period $n + L - j + 1$, i.e., $(l_{n+L-j+1}^b)$, before any unit backlogged during this period.

The right-hand side of (4) involves only aggregate inventory levels and order sizes, undifferentiated by shelf ages or backlog durations. Thus the sample path identity (4) is useful in establishing an equivalence with a traditional level-dependent inventory cost structure by charging the marginal expected costs $\{\beta_{n+L}(j) \in B_{n+L-j} : j = 1, \ldots, n + L\}$ backward to a period prior to $(n + L)$. In determining which period to charge the expected costs to, consider the following: since each of the expressions in (4) involves $q_n$, the order size determined in period $n$, the above expected marginal costs may be charged back no further than period $n$, because at the beginning of any earlier period, $q_n$ is undeterminable and dependent on events yet to come. Indeed, we will show that (4) allows us to charge the above expected marginal backlogging costs backward to period $n$, but only for backlog durations $j \leq L + 1$. When $j \geq L + 2$, (4) involves demands that have already materialized prior to period $n$ so that, in general, charging the marginal cost $\beta_{n+L}(j) \in B_{n+L-j}$ to the beginning of period $n$ would require the optimal order size of period $n$ to potentially depend on the history of past and observed demands rather than the inventory position only. Similar reasoning, charging the expected marginal backlogging costs to a period later than period $n$ aggravates this problem for even more age classes. Thus, it is most advantageous to charge the expected costs to period $n$.

When charging the expected costs $\beta_{n+L}(j) \in B_{n+L-j}$ backward to period $n$, it is therefore necessary to restrict oneself to settings where there is no need to differentiate among backlog durations $j \geq L + 2$ either by (i) restricting the policy space so as to ensure that no demand unit is backlogged for more than $(L + 1)$ periods or (ii) restricting the shape of the $\beta_n(\cdot)$ functions so that they are constant for $j \geq L + 2$. This allows us to deal with the aggregate “senior” block $\sum_{j=2}^{n+L} B_{n+L-j}$ only, and the latter satisfies the sample path identity

$$\sum_{j=L+2}^{n+L} B_{n+L-j} = (y_n)^-.$$  

(5)

If $y_n < 0$, there are $(y_n)^-$ units backlogged at the beginning of period $n$ and hence at the end of period $n - 1$. Since these will not be filled by the end of period $n + L$, they are the demand units that are backlogged at the end of period $n + L$ with a delay of at least $L + 2$ periods. If $y_n \geq 0$, i.e., $(y_n)^- = 0$, all units backlogged at the beginning of period $n$
will be filled at the end of period \( n + L \), so any backlog at the end of that period has a delay at most equal to \( L + 1 \). Thus, in contrast to shelf-age-dependent carrying costs, one of the following two assumptions is required:

**Assumption (NIP) (Nonnegative Inventory Position After Ordering).** We restrict ourselves to policies under which \( y_n \geq 0 \) for \( n = 1, \ldots, N \).

**Assumption (CBL) (Marginal Cost Rate is Constant Beyond Leadtime).** \( \beta_n(j) = \beta(L + 2) \) for any \( n \) and \( j \geq L + 2 \).

These assumptions impose significant restrictions, either on the shape of the backlogging cost rate function (CBL) or by imposing a potentially restrictive constraint for the feasible order set (NIP).

Under Assumption (NIP), \( (y_n)^{-} = 0 \) so that, by (5), no demand units are backlogged for more than \( L + 1 \) periods. Under Assumption (CBL), there is no need to differentiate among backlogs of different ages longer than \( L + 1 \). Thus, either (CBL) or (NIP) implies

\[
\sum_{j=L+2}^{n+L} \beta_{n+L}(j)B_{n+L,j} = \beta(L + 2)(y_n)^{-}.
\]

It thus follows from (4) that the expected backlogging costs charged in period \( n \) are

\[
G^b_n = \sum_{i=1}^{n+L} \beta_{n+L}(j) \mathbb{E} B_{n+L,j}
\]

\[
= \sum_{j=1}^{M+1} \beta_{n+L}(j) \left[ I_{n+L-j+1}(D_{n+L-j+1}, I_{n+L-j+1}) - \left( \sum_{i=n-j+2}^{n} q_i - (I_{n+L-j+1})^{-} \right)^{+} \right]^{+} + \beta(L + 2)(y_n)^{-}.
\]

Assumption (CBL) was identified by Huh et al. (2011, Assumption 5) for the standard periodic review model with full backlogging, a stationary demand distribution and stationary cost parameters, fixed plus linear order costs, and linear holding costs. The authors show that under Assumption (CBL), an \((s, S)\) policy, acting on the inventory position, is optimal. (See also Theorem 2(e) below for the same result on this and more general models.) As to (NIP), Huh et al. (2011) recognize it as an alternative enabler of their cost transformation approach in their model’s special case where there are no fixed costs and \( y_n \geq 0 \) is guaranteed without loss of generality.

### 3.1. Full Backlogging

Lemma 1 and (7) show that the expected age and delay-dependent costs over the planning horizon may be written as a function of the sequence of inventory levels and orders “in the pipeline” only. This, we show, substantially reduces the state space dimension of any dynamic programming formulation. Under full backlogging, all inventory information can be aggregated into the single inventory position measure, similar to what is the case under traditional level-dependent inventory cost structures. Full backlogging implies \( l(d, i) = (d - i)^+ \) and the well-known recursion

\[
l^b_{n+L} = y_n - D[n, n + L - 1].
\]

**Theorem 1 (Full Backlogging: Equivalence with Level-Dependent Carrying Cost).** The model with shelf-age and delay-dependent inventory costs is equivalent to one in which the costs in periods \( n = 1, \ldots, N \) are represented by \( G^b_n(y_n) + G^p_n(y_n) \), a level-dependent convex holding and backlogging cost function of period \( n \)’s inventory position, with \( G^b_n(\cdot) \) given by

\[
G^b_n(y_n) = \sum_{j=1}^{n+L-n} \rho^n y^n - (\alpha_n(j) - \alpha_{n+1}(j-1)) \mathbb{E}(y_n - D[n, j + n + L - 1])^{+},
\]

and, under Assumption (NIP) or (CBL), \( G^p_n(\cdot) \) given by

\[
G^p_n(y_n) = \sum_{j=1}^{L+1} \left[ \beta_{n+L-j+1}(j) - \beta_{n+L-j+2}(j-1) \right]^{+}
\]

\[
\mathbb{E}(D[n, n + L - j + 1] - y_n)^{+} + \beta(L + 2)(y_n)^{-}.
\]

Theorem 1 enables us to transform a model with general shelf-age and delay-dependent costs into an equivalent one in which the system state is one-dimensional and specified by the inventory position only. Thus, regardless of any other model assumptions, any such model is computationally tractable. Moreover, the equivalency allows for rapid derivation of the optimal policy in many inventory systems with general shelf-age and delay-dependent costs. We cover a few basic models.

**Theorem 2 (Structural Results with Full Backlogging).** (a) Assume \( N < \infty \) and order costs are linear; with arbitrary time-dependent order cost rates \( \{c_n, n = 1, 2, \ldots, N\} \) and an arbitrary discount factor \( \rho \leq 1 \). Under Assumption (NIP) or (CBL), a time-dependent base-stock policy is optimal; i.e., in each period, there exists a base-stock level \( S_n^* \), such that it is optimal to raise the inventory position to \( S_n^* \) whenever it is below \( S_n^* \).

(b) Let \( N < \infty \). Assume order costs are linear, with a stationary variable cost rate \( c \) per unit ordered. Assume also that future costs are not discounted; i.e., \( \rho = 1 \). (When \( N = \infty \), we employ the long-run average cost criterion.) A possibly time-dependent base-stock policy is optimal.

(c) In the model of part (a), assume all parameters are stationary and any inventory at the end of the horizon can be returned at the original purchase price. Then \( S_1^* \leq S_2^* \leq \cdots \leq S_N^* \).
(d) Assume $N = \infty$ and the model of part (a), with all parameters stationary. Assume orders are subject to a capacity limit $C$. Under Assumption (CBL)\(^3\) and either the discounted or long-run average cost criterion, there exists a stationary modified base-stock policy with base-stock level $S^*$ that is optimal: if the stationary inventory position in any given period is below $S^*$, an order is placed to bring the inventory position as close as possible to $S^*$.

(e) Assume an order in period $n$ incurs a fixed cost $K_n$ and a variable per unit cost $c_n$. If $N < \infty$, assume $K_n \geq \rho K_{n+1}$ for all $n = 1, \ldots, N-1$. Under Assumption (NIP) or (CBL), there exists a time-dependent $(s_n, s_n)$ policy that is optimal when $N < \infty$. If $N = \infty$, an $(s^*, S^*)$ policy is optimal under either (NIP) or (CBL), both under the discounted total cost and the long-run average cost criteria, assuming all cost parameters and demand distributions are stationary.

(f) Let $N = \infty$ and consider the model with fixed-plus-linear order costs. Assume that all cost parameters, the cost functions $G^L_n(\cdot)$ and $G^H_n(\cdot)$, and the one-period demand distribution depend on a state of the world variable $\omega$ that evolves in accordance with a finite state irreducible Markov chain. Both under the total discounted and the long-run average cost criteria, a state-dependent $(s, S)$ policy is optimal under Assumption (NIP) or (CBL).

To our knowledge, the first and only structural result for models with general shelf-age-dependent carrying costs was obtained in a recent paper of Gupta and Wang (2009). Assuming linear backlogging costs, these authors proved part (a) of Theorem 2 for the case where the $\alpha_n(\cdot)$ functions become flat after a certain period $k$; i.e., $\alpha(j) = \alpha(k)$, for all $j \geq k$. This permits a finite-dimensional state representation when disaggregating inventory levels according to the items’ shelf age. (The state of the system has dimension $k+L$, independent of the planning horizon $N$.) By the equivalency result in Theorem 1, the problem can be formulated as a dynamic program with a one-dimensional state space of a well-known structure; this equivalency holds for arbitrary (increasing) $\alpha_n(\cdot)$ functions.

In their concluding section, Gupta and Wang (2009) question how the inclusion of fixed costs would impact the structure of the optimal policy. Their question is resolved by part (d) of the above theorem; i.e., it is optimal to use an $(s, S)$ policy acting on the inventory position (under the same assumption of stationary cost parameters and demand distributions that is required in the standard model with linear holding costs). Maddah et al. (2004) consider the long-run average cost criterion for the special case of a two-part credit scheme; i.e., the stationary $\alpha(\cdot)$ function adopts two distinct values. These authors restrict themselves to the class of $(s, S)$ policies and develop heuristics to compute the best parameter pair. However, the equivalency results in Theorem 2 show that this class of policies, in fact, contains the optimal policy and that the exact algorithms by Veinott and Wagner (1965) or Zheng and Federgruen (1991) can be used, employing the transformed cost functions $G^L(\cdot) + G^H(\cdot)$.

Part (b) of Theorem 2 was shown by Huh et al. (2011) for general backlogging costs but linear holding costs. (These authors present this result in Remark 4, albeit for a model without any order costs; the generalization to stationary linear order costs is straightforward.) For models with fixed-plus-linear order costs, Huh et al. (2011) establish the optimality of $(s, S)$ policies under assumption (CBL) in combination with another assumption that restricts the shape of the demand distributions. (See their Assumption 2; alternative (a) of this assumption, which restricts the class of demand distributions, permits more general incremental cost rates $\beta(\cdot)$ that are unimodal as opposed to increasing; alternatives (b) and (c) allow for fixed costs in any period with a backlog.)

The extension in part (f) has many important applications, for example, conditions and product demands that are sensitive to underlying economic conditions and demands that fluctuate as a function of the stage in the product life cycle or the number of advance customer orders; see, e.g., Benjaafar et al. (2011) and Gayon et al. (2009).

We identify another interesting equivalency for the model with general shelf-age dependent holding costs but standard level dependent backlogging costs. In this case, the model is equivalent not only to one with level-dependent holding costs but even to one with linear holding costs and an extended (stochastic) leadtime.

**Theorem 3 (Equivalence of Shelf-Age-Dependent and Linear Holding Costs).** Let $N = \infty$ and $\alpha_n(\cdot) = \alpha(\cdot)$ for all $n = 1, 2, \ldots$. Assume $h \equiv \sum_{j=0}^{\infty} \rho [\alpha(j+1) - \alpha(j)] = \rho \lim_{t \to \infty} \alpha(t) < \infty$ and backlogging costs described by a convex function $G^L(y)$. Define a random leadtime $\Lambda$ with the following distribution:

$$P(\Lambda = j) = \frac{\rho [\alpha(j+1) - \alpha(j)]}{h}, \quad j = 0, 1, \ldots \quad (11)$$

The model with shelf-age-dependent inventory carrying costs and level-dependent backlogging costs is equivalent to one with linear holding costs at a constant rate $h$ but an extended leadtime $L + \Lambda$.

When the nonlinear shelf-age-dependent cost structure arises because of trade credit arrangements (see the introduction), the function $\alpha(\cdot)$ is typically piecewise constant. A frequently used structure, referred to as a two-part credit scheme (see Cunat 2007), has an interest-free grace period (F) followed by a constant positive interest rate thereafter. In the infinite horizon model of Theorem 3, a piecewise constant $\alpha(\cdot)$ function is equivalent to linear holding costs with an additional leadtime component $\Lambda$ that has support only on the period lengths in which the function $\alpha(\cdot)$ experiences an upward jump; see (11). For the two-part credit scheme as an example, the additional leadtime is simply $F$ because it is the only period in which the function $\alpha(\cdot)$ has an upward jump.
3.2. Systems with Partial Backlogging or Lost Sales

Consider now a system in which no or only a part of unsatisfied demand is backordered, under general shelf-age and delay-dependent inventory costs and positive leadtimes. Even under the simplest (e.g., linear) holding and backlogging cost functions, it is necessary to represent the system via an \((L + 1)\)-dimensional state vector consisting of the inventory level and the orders placed in the prior \(L\) periods. Our analysis reveals that under general age and delay-dependent costs, the state does not need to be expanded beyond this \((L + 1)\)-dimensional vector.

Consider, first, the case where \(L = 0\). Since \(I_n^b = y_n\), \(G^b_n(\cdot)\) in (3) is a convex function of \(y_n\) and gives the expected holding costs. Under (NIP) or (CBL), expression (7) for the expected backlogging costs reduces to

\[
G^b_n(y_n) = \beta_n(1)E I_n(D_n, y_n) + \beta(2)(y_n)^-.
\]

The structural properties of the function \(G^b_n(y_n)\) depend, of course, on those pertaining to the functions \(I_n(d, i)\). The simplest backlogging structure has a given fraction \(r < 1\) willing to be backlogged; i.e., \(I_n(d, i) = r(d - i)^+\). (When \(r = 0\), we obtain the lost sales case.) Thus,

\[
G^b_n(y_n) = r \beta_n(1)(D_n - (y_n)^+) + \beta(2)(y_n)^-
\]

is a piecewise linear and convex function of \(y_n\) since \(\beta(2) \geq \beta_n(1)\). (Note that the function \(I_n(d, i)\) itself fails to be convex in \(i\).) This gives rise to the following theorem:

**Theorem 4.** Consider an inventory system with partial backlogging or lost sales and \(L = 0\).

(a) The model with shelf-age and delay-dependent inventory costs is equivalent to one in which, in each period \(n = 1, \ldots, N\), an inventory level (or position) dependent function \(G^b_n(\cdot) \equiv G^b_n(\cdot) + G^b_n(\cdot)\) is charged with \(G^b_n(\cdot)\) given by (3) and \(G^b_n(\cdot)\) given by (12) under Assumption (NIP) or (CBL).

(b) Under the partial backlogging structure \(I_n(d, i) = r(d - i)^+\), all of the functions \(G^b_n(\cdot)\) are convex and all of the results of Theorem 2 continue to apply.

When \(L > 0\), (3) shows that all expected carrying costs can be assessed as functions of the aggregate beginning inventory levels only, i.e., without having to decompose these inventory levels by shelf age. It suffices to charge to period \((n + L)\) the function \(G^b_n(I_{n}^{b+L})\), which depends on that period’s starting inventory only. As to delay-dependent backlogging costs, note the recursion

\[
I_{n+1}^b = (I_n^b + q_{n-L} - D_n)^+ - I(D_n, I_n^b + q_{n-L}).
\]

Thus, by repeated substitutions of (14), one verifies that all of the beginning inventory levels \([I_n^b, I_{n+1}^b, \ldots, I_{n+L}^b]\) can be expressed as functions of \([I_n^b, q_{n-1}, \ldots, q_{n-L}]\), the state at the beginning of period \(n\), as well as \([D_n, D_{n+1}, \ldots, D_{n+L}]\), which are unknown at the beginning of this period. By (10), this implies, under either (NIP) or (CBL), that the total expected backlogging costs at the end of period \((n + L)\) can be characterized as a function of this system state. In other words, in spite of the presence of general shelf-age and delay-dependent costs, any model with partial backlogging or lost sales can be solved as a dynamic program with the same state and action spaces as one in which all holding and backlogging costs are linear (or level dependent).

**Remark:** In lost sales models, the structure of the optimal policy is complex, even under the simplest setting with linear order costs. The model was first analyzed by Karlin and Scarf (1958). Recently, Zipkin (2008a, b) identified bounds and monotonicity properties of the optimal order quantities. Although these references assume that a specific convex cost function of each period’s beginning inventory level is charged, it can easily be verified that all results apply to arbitrary convex functions. Thus, all of the bounds and monotonicity results for the standard model with linear holding costs continue to apply under general shelf-age-dependent costs.

4. Continuous Review Models

Continuous review inventory models with discrete demand epochs typically assume that demands are generated by a compound renewal process: the times between demand epochs are i.i.d. random variables \(X_1, X_2, \ldots\) distributed like \(X\), without loss of generality, with mean 1. At each demand epoch, a random quantity is demanded. Let \([Z_n, n = 1, 2, \ldots]\) denote the independent and identically distributed (i.i.d.) sequence of demand quantities, distributed as a random variable \(Z\), with a finite moment generating function in some interval around zero. Orders are placed at demand epochs (only). Our objective is to minimize the long-run average cost by considering the semi-Markov decision process (SMDP) that is embedded in the system on demand epochs. Like all other SMDP, control of the system (i.e., placing orders) is restricted to the time points (i.e., demand epochs) on which the process is embedded. This assumption is without loss of generality when demand follows a Poisson process, but it may cause loss of optimality for other general demand processes, as standard as it is for all SMDP.

In this continuous review model, \(\alpha(t)\) now denotes the marginal inventory cost rate incurred for an item that has a shelf age \(t\) and \(\beta(t)\) the marginal backlogging cost rate when a unit of demand has been waiting for \(t\) time units. In accordance with the stationary version of the (ODY) and (ODY-B) properties, the functions \(\alpha(\cdot)\) and \(\beta(\cdot)\) are, again, assumed to be increasing. In view of the monotonicity of the \(\alpha(\cdot)\) and \(\beta(\cdot)\) functions and because orders do not cross, it continues to be optimal to deplete inventories on a FIFO basis. To ensure that various expected carrying and backlogging costs are finite, we assume that both functions \(\alpha(\cdot)\) and \(\beta(\cdot)\) are polynomially bounded; i.e., there
exists a power $l$ such that $\alpha(t) \leq Mt^l$ and $\beta(t) \leq Mt^l$, for some $M > 0$. In addition, the $(l + 1)$-st moment of $X_1$ is assumed to be finite; i.e., $\mathbb{E}(X_1^{l+1}) < \infty$.

We confine ourselves to models with full backlogging as opposed to our treatment of periodic review models, where we consider lost sales or partial backlogging in parallel with the case of full backlogging. For the sake of brevity, we only address the standard inventory model in which there are constant fixed and variable order costs in conjunction with the above shelf-age-dependent inventory and delay-dependent backlogging costs and in which orders of arbitrary size may be placed at any demand epoch. Orders arrive after a fixed leadtime $L$. (Extensions to stochastic leadtimes arising from exogenous and sequential processes are straightforward.)

In this model, we will show that an $(s, S)$ policy acting on the inventory position is optimal, under Assumption (NIP) or (CBL), and assuming the interdemand distribution $X$ has the New-Better-than-Used (NBU) property, a property weaker than the Increasing Failure Rate (IFR) or even the Increasing Failure Rate Average (IFRA) condition and therefore shared by most commonly used distributions for interdemand times; see below. Under these assumptions, the model is equivalent to a periodic review model whose single period demand is distributed as $Z$ and with a specific convex one-step expected cost function of the inventory position. This equivalency implies that the structural results, obtained in Theorem 2 under alternative assumptions about order costs and capacity limits, can be established in a continuous review model as well.

We first show, in §4.1, that the expected inventory carrying costs over the infinite planning horizon may be represented as a sum of convex functions, where the $n$th term is a function of $y_n$, the inventory position after ordering at the $n$th demand epoch. The same type of representation may be shown for the expected backlogging costs; see §4.2. In §4.3, we use these characterizations to derive the structure of the optimal policy.

### 4.1. Inventory Carrying Cost Transformation

Let $S_n = \sum_{j=1}^n X_j$ the time of the $n$th demand epoch, and let $N(t)$ denote the number of demand epochs in an interval of length $t$ following a renewal epoch. It is tempting to apply the stock decomposition approach in §3, where the expected inventory carrying costs incurred between the $n$th and the $(n + 1)$-st demand unit; i.e., $[S_n + L, S_{n+1} + L]$ are assessed by decomposing the inventory level $I_{S_n+L}$ at time $S_n + L$, according to the shelf age of the various units in $I_{S_n+L}$. In periodic review systems, all units in stock at $S_n + L$ remain in inventory until the end of the interval $[S_n + L, S_{n+1} + L]$. In contrast, under a general compound renewal demand process, any number of demand epochs may arise in the interval. For a given age decomposition of $I_{S_n+L}$, the expected inventory carrying costs in $[S_n + L, S_{n+1} + L]$ depend in a complicated manner on the joint distribution of the demand epochs and demand sizes that arise during that interval as well as the amount of time elapsed since the last demand epoch preceding $S_n + L$.

We are thus compelled to adopt a different representation of the inventory costs. Since every unit that resides in inventory at some point in time is part of a unique order, we assess the total (expected) future carrying costs incurred for all units injected into the system by the placement of an order at a given, say, the $n$th, demand epoch, and sum these costs over all demand epochs and associated orders. Levi et al. (2007, 2008a, b) refer to this as the marginal holding cost accounting approach. Let

$q_n = \text{order size at the } n\text{th demand epoch, } n = 1, 2, \ldots$;

$x_n = \text{the inventory position right after the } n\text{th demand epoch but before placement of an order; }$

$y_n = \text{the inventory position after ordering at the } n\text{th demand epoch; }$

$q_n^0 = \text{the part of the order } q_n \text{ that remains in inventory beyond the order’s delivery, after the clearing of any demand units backlogged at that time; }$

$q_n^j = \text{the part of the order } q_n \text{ that remains in inventory beyond the } j\text{th demand epoch following the order’s delivery, } j = 1, 2, \ldots$.

Consider therefore the $n$th demand epoch, for any $n \geq 1$, at which the inventory position is increased from a level $x_n$ to a level $y_n \geq x_n$. The order arrives $L$ time units later and is sequentially depleted at subsequent demand epochs, depending on the prevailing inventory at the time the order arrives and the demand quantities at subsequent demand epochs. The first demand epoch following the order’s arrival has a distribution $R(L) \equiv \text{the excess renewal distribution at time } L \text{ for a renewal process starting at time } 0$.

All subsequent demand epochs follow this first epoch after an interrenewal time distributed like $X$. In other words, the demand epochs following the order arrival represent a delayed renewal process. Define $\bar{S}_{n,j}$ as the time between the arrival of the order placed at $S_n$ and the $j$th subsequent demand epoch, $j \geq 0$. Clearly, $\bar{S}_{n,0} = 0$. For $j \geq 1$, we have

$$\bar{S}_{n,j} \overset{\diamond}{=} R(L) \oplus \sum_{i=1}^{j-1} X_i, \quad (15)$$

where $\overset{\diamond}{=} \text{ denotes equality in distribution and } \oplus \text{ denotes the convolution operation.}$

Upon delivery, the part of the order that remains after filling any prevailing backlogs is part of the system’s inventory for at least $R(L)$ time units; similarly, the part of the order that is left after filling demands at the $j$th demand epoch following the order’s arrival sees its shelf age increase from $\bar{S}_{n,j}$ to at least $\bar{S}_{n,j+1}$. Each such unit therefore incurs additional carrying costs between the $j$th and $(j + 1)$st demand epochs, given by $H(\bar{S}_{n,j+1}) - H(\bar{S}_{n,j})$, where $H(t) = \int_0^t \alpha(s) \, ds$. Note from (15) that for any given $j \geq 0$, the random variables $\{H(\bar{S}_{n,j+1}) - H(\bar{S}_{n,j}), n \geq 1\}$ are identically distributed and, hence, have a common mean $\eta_j$. 


Lemma 2 below shows that the sequence \{\eta_j, j \geq 0\} is finite and increasing provided that the interrenewal time \(X\) is NBU; i.e., for all \(t \geq 0\), \((X - t) X > t\) \(\leq X\). The NBU property is weaker than the Increasing Failure Rate or even the Increasing Failure Rate Average condition; it is shared by most commonly used distributions such as the Normal, the exponential, the Weibull, the Gamma, and the power distribution.

**Lemma 2.** Assume the interrenewal distribution \(X\) has the NBU property. Define \(\eta_{-1} = 0\). Then there exists a constant \(C > 0\) such that \(0 \leq \eta_{j-1} < \eta_j \leq C j^{\gamma+1}\) for all \(j \geq 0\).

The total expected carrying costs associated with the order placed right after the \(n\)th demand epoch, if any, are thus given by

\[
E \sum_{j=0}^{\infty} (H(\hat{S}_{n,j+1}) - H(\hat{S}_{n,j})) q_n^j, \tag{16}
\]

Define \(I(y) \equiv \sum_{j=0}^{\infty} (\eta_j - \eta_{j-1}) E(y - \sum_{i=1}^{N(L)+j} Z_{n+1})^+\) for any \(n \geq 1\). The value of the function \(I(\cdot)\) is independent of \(n\) because of the sequence of variables \(\{Z_j, j \geq 1\}\) being independent and identically distributed. The following lemma shows that the long-run average holding costs are tightly bounded from below by those arising from a continuous review model where \(I(y)\) is charged at the \(n\)th demand epoch at which the inventory position after ordering is \(y_n\). A key step in the proof is to express each quantity \(q_n^j\) exclusively as a separable function of the inventory positions \(y_n\) and \(y_{n-1}\), and no other inventory on order measures. This is achieved by the following sample path identity: for all \(j \geq 0\),

\[
q_n^j = \left(y_n - \sum_{i=1}^{N(L)+j} Z_{n+i}\right) + \left(y_{n-1} - \sum_{i=0}^{N(L)+j} Z_{n+i}\right). \tag{17}
\]

**Lemma 3.** Assume \(X\) is NBU.

(a) The identity (17) holds almost surely.

(b) The function \(I(\cdot)\) is finite and convex.

(c) For any policy \(\pi \in \Pi\), the long-run average holding costs are bounded from below by those that arise when the cost \(I(y)\) is charged at any demand epoch at which the inventory position after ordering equals \(y\); i.e.,

\[
\lim_{T \to \infty} \frac{1}{T} E \sum_{n=1}^{N(T)} \sum_{j=1}^{\infty} [H(\hat{S}_{n,j+1}) - H(\hat{S}_{n,j})] q_n^j \geq \lim_{T \to \infty} \frac{1}{N(T)} \sum_{n=1}^{\infty} I(y_n), \tag{18}
\]

with equality in (18) for any policy \(\pi\) for which the sequence \(\{y_n\}\) is uniformly bounded from above.

### 4.2. Backlogging Cost Transformation

To account for delay-dependent backlogging costs, we charge at time \(S_n\) the expected total backlogging costs incurred in the interval \([S_n + L, S_{n+1} + L]\), similar to the approach in §3. We show that these expected backlogging costs can be expressed as a convex function of \(y_n\). As in the models in §3, this representation requires an assumption guaranteeing either that no demand unit is delayed by more than \(L\) time units (Assumption NIP) or that the marginal backlogging cost rate is constant for delays exceeding \(L\) (Assumption CBL). We confine ourselves to Assumption (NIP), which is equivalent to imposing the policy constraint \(y_n > 0\). The analysis under Assumption (CBL) is similar.

We decompose the set of demand units that are backlogged during \([S_n + L, S_{n+1} + L]\) in accordance with the times at which the demands occurred. Let \(B_{S_n+L,i}\), with \(j \geq 1\), denote the number of backlogged demand units that arrive at \(S_n + j\) and are backlogged during the interval \([S_n + L, S_{n+1} + L]\). By definition, \(B_{S_n+L,i}\) is nonnegative if \(S_n + j < S_{n+1} + L\) and equals zero otherwise. All units demanded up to time \(S_n\) are filled by time \(S_n + L\) since \(y_n > 0\) by Assumption (NIP). \(y_n\) represents the excess inventory at time \(S_n + L\) to satisfy later demands, i.e., those arriving at time \(S_{n+1}\), \(j \geq 1\). By the FIFO rule,

\[
\sum_{i=1}^{j} B_{S_n+L,i} = \left(\sum_{i=1}^{j} Z_{n+i} - y_n\right)^+, \quad j \geq 1 \text{ such that } S_n + j < S_{n+1} + L. \tag{19}
\]

Let \(\Delta j_{n}\) denote the incremental backlogging costs each unit in \(B_{S_n+L,i}\) incurs during \([S_n + L, S_{n+1} + L]\). The total incremental backlogging costs during \([S_n + L, S_{n+1} + L]\) are given by \(\sum_{j=1}^{\infty} \Delta j_{n} B_{S_n+L,i}\). Lemma 4 below shows that the expectation of this quantity is a convex function of \(y_n\) only.

**Lemma 4.** Assume \(X\) is NBU.

(a) For \(j = 1, 2, \ldots\), the sequence \(\{\Delta j_{n}\}\) has identical mean \(\xi_j\), and \(0 < 0 < \xi_j < \infty\).

(b) There exists a constant \(A > 0\) and \(0 < r < 1\) such that \(\xi_j \leq A r^{j-1}\), \(j = 1, 2, \ldots\).

(c) The total expected backlogging costs charged to the \(n\)th demand epoch is given by \(B(y_n) \equiv \sum_{j=1}^{\infty} (\xi_j - \xi_{j+1}) E(\sum_{i=1}^{j} Z_{n+i} - y_n)^+ < \infty\). The function \(B(\cdot)\) is convex.

### 4.3. The Optimal Inventory Policy

In §§4.1 and 4.2 we have shown that under an arbitrary policy \(\pi\), the expected costs charged in any finite time interval \([0, T]\) may be expressed solely as a function of the sequence of inventory positions \(\{x_n, y_n\} : n = 1, 2, \ldots\), and the state of the system dynamics is completely determined by this sequence as well: \(x_{n+1} = y_n - Z_{n+1}\).

We now show that an \((s, S)\) policy acting on the inventory position is optimal by establishing that our continuous review model is equivalent to the following standard periodic review model:

**Periodic Review Model (PR):** when a period starts with an inventory position \(x\), an order may be placed to elevate the inventory position to any level \(y \geq (x)^+\). The
immediate expected costs incurred during this period are given by \( G^n(x, y) = K\delta(y - x) + c(y - x) + I(y) + B(y) \). Demands are i.i.d. random variables distributed like \( Z \). Orders arrive after a leadtime of \( L \) periods and all stockouts are backlogged.

**Theorem 5 (Compound Renewal Demand Processes).** Assume \( X \) is NBU and the restriction (NIP) holds. The original continuous review model is equivalent to the Periodic Review Model (PR); the same \((s, S)\) policy is optimal in both.

An additional implication of the equivalency between our continuous review model with the periodic review model with standard level dependent cost functions is that the optimal policy values \( s^* \) and \( S^* \) may be computed by determining the optimal policy parameter values \( s^* \) and \( S^* \) in the PR model (see, e.g., Zheng and Federgruen 1991). Similarly, well-known bounds for \( s^* \) and \( S^* \) may be invoked directly (see, e.g., Veinott and Wagner 1965).

### 5. Conclusion

Motivated by a variety of applications discussed in §1, we have addressed periodic and continuous review models with general shelf-age and delay-dependent inventory costs. We have shown that such a model may be transformed into an equivalent model with level-dependent holding and backlogging costs. The equivalence results allow us to characterize the structure of an optimal procurement strategy and to “adopt” known algorithms to compute such optimal strategies.

Under full backlogging, the equivalent model has a one-dimensional state space with the inventory position as the state of the system; under more complex stockout dynamics (lost sales or partial backlogging), it is necessary to keep track of the inventory level and the complete pipeline of outstanding orders, but this \((L + 1)\)-dimensional state space is identical to what is required under the simplest, i.e., linear, holding and backlogging costs. Our assumption that the marginal cost rates increase with an item’s shelf age or a demand unit’s delay duration implies convexity properties of the equivalent level-dependent cost functions, which in turn help guarantee various structures for the optimal strategy; see Theorems 2, 4, and 5.

As explained at the end of §1, there exist fundamental differences between the cost transformation approaches required for shelf-age-dependent carrying costs as opposed to delay-dependent backlogging costs. Finally, a model with general shelf-age-dependent costs and full backlogging is shown to be equivalent to one with linear costs but an extended stochastic leadtime.

### Supplemental Material

Supplemental material to this paper is available at http://dx.doi.org/10.1287/opre.2015.1369.

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### Appendix. Proofs

**Proof of Lemma 1:** The total expected shelf-age-dependent holding costs incurred in period \( n + L \) is given by

\[
\sum_{j=1}^{n+L} \alpha_{n+1-j}(j) \mathbb{E}_w I_{n+L,j} = \sum_{j=1}^{n+L} (\alpha_{n+1-j}(j) - \alpha_{n+2-j}(j-1)) \sum_{w=j}^{n+L} \mathbb{E}_w I_{n+L,w} \]

\[
= \sum_{j=1}^{n+L} (\alpha_{n+1-j}(j) - \alpha_{n+2-j}(j-1)) \cdot \mathbb{E}_w \{I_{n+L-1-j}^p - D(n + L + 1 - j, n + L)\}^+, \]

where the first equality follows from \( \alpha_0(j) = 0 \) for all \( j \) and the second one from (1). Hence, the total inventory carrying cost during the planning horizon, discounted back to period 1, is given by

\[
\sum_{n=1}^N \rho^{nL} \sum_{j=1}^{n+L} (\alpha_{n+1-j}(j) - \alpha_{n+2-j}(j-1)) \cdot \mathbb{E}_w \{I_{n+L-1-j}^p - D(w + L, n + L)\}^+ \]

\[
= \sum_{n=1}^N \rho^{nL} \sum_{w=1}^{n} \alpha_w(n + 1 - w) - \alpha_{w+1}(n - w)) \cdot \mathbb{E}_w \{I_{n+L-1-j}^p - D(w + L, n + L)\}^+ + H_0(I_1^p, \ldots, I_n^p) \]

\[
= \sum_{n=1}^N \sum_{w=1}^{n} \rho^{nL} \alpha_w(n + 1 - w) - \alpha_{w+1}(n - w)) \cdot \mathbb{E}_w \{I_{n+L-1-j}^p - D(w + L, n + L)\}^+ + H_0(I_1^p, \ldots, I_n^p) \]

\[
= \sum_{n=1}^N \sum_{j=1}^{n} \rho^{n+L-1} (\alpha_w(j) - \alpha_{w+1}(j - 1)) \cdot \mathbb{E}_w \{I_{n+L-1-j}^p - D(w + L, j + w + L - 1)\}^+ + H_0(I_1^p, \ldots, I_n^p), \]

where \( H_0(I_1^p, \ldots, I_n^p) = \sum_{k=1}^n \rho^{n+L} \sum_{w=1}^{n} \alpha_w(n + 1 - w) - \alpha_{w+1}(n - w)\} \mathbb{E}_w \{I_{n+L-1-j}^p - D(w + L, n + L)\}^+ \). The convexity of \( G^n(\cdot) \) follows from \( \alpha_w(j) - \alpha_{w+1}(j - 1) \geq 0 \) by (ODY) and because each of the terms \( \mathbb{E}_w \{I_{n+L-1-j}^p - D(w + L, j + w + L - 1)\}^+ \) is convex in \( I_{n+L}^p \). \( \square \)

**Proof of Theorem 1:** Using the identity (8), (9) easily follows from (3). To consider the total backlogging costs, note first that (4) is reduced to a simpler form under full backlogging:

\[
B_{n+L,j} = \left( D_{n+L-j+1} - (I_{n+L-j}^p)^+ \right)^+ - \left( \sum_{l=n-j+2}^n q_l - (I_{n+L-j}^p)^+ \right)^+ \]
\[
D_{n+L-j+1} = \left( \sum_{i=n+1}^{n+j} q_i - (i_{n+L-j+1}) \right) + \\
(D_{n+L-j+1} - (n - j + n - L)) + \\
(D_{n+L-j+1} - (n + L - j)) + \\
(D_{n+L-j+1} - (n + L)) + \\
(D_{n+L-j+1} - (n + n - L)) + \\
(D_{n+L-j+1} - (n + n - L - j)) + \\
(D_{n+L-j+1} - (n + n - L - j - y)) - (n + n - L - j - y). 
\]

At the right-hand side of the second equation sign above, we took the positive part operator \( + \) away from the expression \( (D_{n+L-j+1} - (i_{n+L-j+1})) + \) and the equality still holds since both sides of the equation equal zero when that expression is negative. The third equality can easily be verified by considering, separately, the case \( i_{n+L-j+1} > 0 \) and \( i_{n+L-j+1} \leq 0 \). The fourth equation follows from (8), and the fifth one follows from the well-known recursion \( y_n = y_{n-j} + \sum_{i=n+1}^{n+j} q_i - D[n - j + 1, n - 2] \). The last equality may easily be verified by distinguishing between the case where \( y_n \) is greater than \( D[n, n + L - j] \) and the case where it is smaller.

Under Assumption (NIP) or (CBL), it follows from (7) that the total backlogging costs at the end of period \( n + L \) are given by

\[
G_n = \sum_{j=1}^{L-1} \beta_{n+L-j+1}(j) [(D[n, n + L - j + 1] - y)_+ \\
- (D[n, n + L - j] - y)_+] + \beta(L + 2)(y)_- \\
= \sum_{j=1}^{L-1} \left( \beta_{n+L-j+1}(j - \beta_{n+L-j+1}(j-1)) \\
\cdot (D[n, n + L - j + 1] - y)_+ + \beta(L + 2)(y)_- 
\right)
\]
in view of the convention of \( \beta_j(0) = 0 \) for any \( j \), which proves (10).

**Proof of Theorem 2:** (a) Under Assumption (CBL), the result follows from Scarf (1960) by invoking the equivalency result in Theorem 1.

Under the (NIP) assumption, the nonnegativity constraints for the action variables \( y_n \) may be handled as follows: first consider the relaxed problem where the constraint \( y_n \geq 0 \) is relaxed. By the equivalency result in Theorem 1, a time-dependent base-stock policy is optimal in the “equivalent” model. If \( S_n^* \geq 0 \) for all \( n = 1, \ldots, N \), this policy satisfies the relaxed constraint and is therefore optimal in the original problem as well. Otherwise, transform the one-step expected backlogging cost functions \( G_n(\cdot) = G_n^b(\cdot) + G_n^g(\cdot) \) to functions \( G_n(\cdot | M) \) defined as follows: \( G_n(\cdot | M) = G_n(\cdot) \) for \( y \geq 0 \) and \( G_n(\cdot | M) = \exp(-\lambda y) + G_n(0) - 1 \) for \( y < 0 \). Note that the functions \( G_n(\cdot | M) \) continue to be convex for \( M \) sufficiently large, so a time-dependent base-stock policy continues to be optimal. Moreover, for \( M \) sufficiently large, say \( M \geq M_1 \), \( S_n^*(M) \geq 0 \) for all \( n \). Finally, \( S_n^*(M) = S_n^*(M_1) \) for all \( M \geq M_1 \) and all \( n = 1, \ldots, N \) since the value of the total expected costs is independent of \( M \) as long as \( S_n^* \geq 0 \).

(b) In this part, it is easily verified that Assumption (NIP) holds, without loss of optimality: backlogs must be cleared ultimately and there is no incentive to delay the clearance of any previously backlogged demand at the beginning of any period; thus \( y_n \geq 0 \), without loss of optimally. The optimality of time-dependent base-stock policy follows immediately from part (a).

(c) Since all cost parameters are stationary, we omit their subscriptions in the proof. Note that the total expected cost over the finite planning horizon may be written as

\[
\sum_{n=1}^{N} \rho^n \left[ c(y_n - x_n - G_n^b(y_n)) + G_n^g(y_n) \right] = \rho^{N+1} c E[y_N - D[N, N + L] + \sum_{n=1}^{N} \rho^n \left[ c(1 - \rho)y_n + G_n^b(y_n) + G_n^g(y_n) \right] - p \rho \sum_{n=1}^{N} (\rho^n + \rho^{N+1}) E[D]
\]

Let \( L_a(y) \equiv c(1 - \rho)y + G_n^b(y) + G_n^g(y) \), a convex function by Theorem 1. Let \( S_n^* = \arg \min L_a(y) \), defined as the smallest minimizer of the function. We will show that \( S_n^* \leq S_n^* \leq \cdots \leq S_N^* \). This implies the optimality of the following myopic policy in each period \( n \): adopt the base-stock policy with level \( S_n^* \). The ordering decisions prescribed by this policy optimize each of the \( N \) terms separately and therefore the aggregate expression as well. We provide the proof assuming inventory levels vary continuously; the case where the demand distribution, and hence the inventory levels, are discrete can be handled in a similar way.

With continuous inventory levels, it suffices to show that \( L_a(\cdot) \geq L_{n+1}(\cdot) \) for all \( n \), which is equivalent to showing that

\[
(G_n^b(y) + G_n^g(y))' = \sum_{j=1}^{L-1} \rho^j [\alpha(j - \alpha(j - 1)) P(D[j, j + n + L - 1] \leq y) \\
- \sum_{j=1}^{L+1} \beta(j - \beta(j - 1)) P(D[j, n + L - 1] \leq y) \\
- \sum_{j=1}^{N-n+1} \rho^j [\alpha(j - \alpha(j - 1)) P(D[1, j + L] \leq y) \\
- \sum_{j=1}^{L+1} \beta(j - \beta(j - 1)) P(D[1, n + L - 1] \geq y) \\
- \sum_{j=1}^{N-n+1} \rho^j [\alpha(j - \alpha(j - 1)) P(D[1, n + L - 1] \geq y) \\
- \sum_{j=1}^{L+1} \beta(j - \beta(j - 1)) P(D[1, n + L - 1] \geq y) \\
= G_{n+1}(y) + G_{n+1}^b(y),
\]

where the inequality follows because the first term on the right side equals the first term on the left side plus one extra positive term, whereas the second terms are the same.
Proof of Lemma 2: which proves the claim.

Proof of Theorem 3: When $N = \infty$ and $\alpha_n(\cdot) = \alpha(\cdot)$, (9) becomes

\[
G_n^\pi(y_n) = \sum_{j=0}^{\infty} \rho^{j} \int_0^\infty \alpha(j) \Phi(y_n - D(n, L + n + j)) \, dt
\]

which follows from the simple dynamics $x_n = y_{n-1} - Z_n$.

Proof of Lemma 3:

(a) For all $j \geq 0$,

\[
q'_n = \left( q_n - \left( \sum_{i=0}^{N(\cdot)_{j}} Z_{n+i} - x_n \right) \right)^+ 
\]

(b) Note that

\[
\mathbb{E} \left[ y - \sum_{i=1}^{N(T)_{j}} Z_i \right]^+ \leq \mathbb{E} \left[ y - \sum_{i=1}^{j} Z_i \right]^+ = \mathbb{E} \left[ \sum_{i=1}^{j} Z_i \right]^+ 
\]

where the first inequality follows from the simple dynamics $x_n = y_{n-1} - Z_n$, which is independent of the renewal process of demand epochs $S_j$, which is independent of both the history of the renewal process proceeding $S_j$ as well as the demand size process $Z_i$, $i \geq 1$, and hence of $q'_n$, which by (17), via $y_n$ and $y_{n-1}$ only depends on $S_1, S_2, \ldots, S_j$ and $Z_i$, $i \geq 1$. Substituting (17) into (22) we obtain the total expected carrying costs associated with $q_n$ as given by

\[
\sum_{j=0}^{\infty} \eta_j \mathbb{E} q'_n 
\]

where the first inequality follows from the convexity of $H(\cdot)$ and $X_{j-1} \geq 0$ almost surely, and the third equality follows from $X_j$ and $X_{j-1}$ being identically distributed.

Proof of Lemma 3: (a) For all $j \geq 0$,

\[
q'_n = \left( q_n - \left( \sum_{i=0}^{N(\cdot)_{j}} Z_{n+i} - x_n \right) \right)^+ 
\]

(b) Note that

\[
\mathbb{E} \left[ y - \sum_{i=1}^{N(T)_{j}} Z_i \right]^+ \leq \mathbb{E} \left[ y - \sum_{i=1}^{j} Z_i \right]^+ = \mathbb{E} \left[ \sum_{i=1}^{j} Z_i \right]^+ 
\]

where the first inequality follows from the simple dynamics $x_n = y_{n-1} - Z_n$.
The long-run average holding costs, under an inventory Markov policy $\pi$, are given by

$$
\lim_{T \to \infty} \frac{1}{T} \sum_{n=1}^{N(T)} \sum_{j=0}^{\infty} \eta_j E q_n^j = \lim_{T \to \infty} \frac{1}{N(T)} \sum_{n=1}^{N(T)} \sum_{j=0}^{\infty} \eta_j E q_n^j
$$

$$
\begin{align*}
&= \lim_{T \to \infty} \frac{N(T)}{T} \sum_{n=1}^{N(T)} \sum_{j=0}^{\infty} \eta_j E \left( y_n - \sum_{i=1}^{N(L)+j} Z_{N(i+)}^n \right) \\
&+ \lim_{T \to \infty} \frac{1}{T} \sum_{j=0}^{\infty} \eta_j E \left( y_N(T) - \sum_{i=1}^{N(L)+j} Z_{N(T)+i} \right) \\
&- \lim_{T \to \infty} \frac{1}{T} \sum_{j=0}^{\infty} \eta_j E x_i - \sum_{i=1}^{N(L)+j} Z_{i+1}^n \right)
\end{align*}
$$

$$
\geq \lim_{T \to \infty} \frac{1}{N(T)} \sum_{n=1}^{N(T)} E I(y_n).
$$

The equality follows from (23). To verify the inequality that in the first term to the left of the inequality, $\lim_{T \to \infty} (N(T)/T) = 1$, almost surely, by the basic renewal theorem and the standardization $E[X] = \tau = 1$, and the second term to the left of the inequality is nonnegative, by Lemma 2. In addition, by an argument similar to that of part (b)

$$
\sum_{j=0}^{\infty} \eta_j E \left( x_i - \sum_{i=1}^{N(L)+j} Z_{i+1}^n \right) < \infty,
$$

which explains why the the third term to the right of (24) vanishes.

Finally, consider a policy $\pi$ under which the sequence $\{y_n\}$ is uniformly bounded from above. In that case, equality in (18) can be proved by showing that the second term to the right of (24) vanishes, just like the third term. This can, again, be verified by showing that

$$
\sum_{j=0}^{\infty} \eta_j E \left( y_N(T) - \sum_{i=1}^{N(L)+j} Z_{N(T)+i} \right) < \infty.
$$

The proof of (26) is analogous to that of part (b), noting that $E[Y_N(T)e^{i\lambda T}] < \infty$ since $\{y_n\}$ is uniformly bounded. □

**Proof of Lemma 4:** (a) We first show the following equations for any $j > 1$:

$$
\Delta J_{j,n} = \int_{(S_{n+1}+L-S_{n+j})^+}^{(S_{n+1}+L-S_{n+1})^+} \beta(u) du
$$

$$
\equiv \int_{(L-(x_{n+j}+\cdots+x_{n+1})^+)}^{(L-(x_{n+1}+\cdots+x_{n+1})^+)} \beta(u) du.
$$

(27)

The second equality follows immediately from the definition of $S_n$. To prove the first equality, we discuss two cases depending on the ordering of the magnitudes of $S_{n+j}$ and $S_{n+1}$. For $j$ such that $S_{n+j} < S_{n+1}$, the backlogged demand units in $B_{S_{n+1}+L}$ already exist at $S_{n+1}$. All those units remain unfilled at least until the next order arrival time $S_{n+1} + L$. Therefore, during the interval $[S_{n+1} + L, S_{n+1} + L)$, every unit in $B_{S_{n+1}+L}$, since demanded at time $S_{n+1}$, sees its delay value increase from $S_{n+1} + L - S_{n+j}$ to $S_{n+1} + L - S_{n+1}$, which proves (27). In the second case where $S_{n+j} > S_{n+1}$, the backlogged demand units in $B_{S_{n+1}+L}$ arrive during the interval $[S_{n+1} + L, S_{n+1} + L]$. Similarly, all those units remain unfilled at least until the next order arrival time $S_{n+1} + L$ and their delay value increases from 0 to $S_{n+1} + L - S_{n+j}$ during the interval $[S_{n+1} + L, S_{n+1} + L]$. Therefore, the incremental backlogging costs associated with each of those units are

$$
\int_{0}^{S_{n+1}+L-S_{n+j}} \beta(u) du,
$$

(28)

which shows that (27) holds in the second case, too. (Note that the expression for $\Delta J_{j,n}$ applies only to $j$ such that $S_{n+j} < S_{n+1} + L$ discussed in the above two cases but also when $S_{n+j} > S_{n+1} + L$ since, in that case, $\Delta J_{j,n} = 0$.)

By the properties of the compound renewal process, we have, for any $j > 1$, that the sequence of random variables $\{\Delta J_{j,n}; n = 1, 2, \ldots\}$ is identically distributed. To show $\zeta_j \equiv \xi_{j+1}$ for any $j > 1$, it suffices to show that almost surely

$$
\Delta J_{j,n} \geq \Delta J_{j+1,n},
$$

(29)

since $\beta(t)$ is increasing and the limits of integration on the left side are greater than their counterparts on the right side, it suffices to prove the width of the integration interval on the left side is larger than that on the right side; i.e.,

$$
(a_2)^+ - (a_1)^+ \geq (a_2 - X_{n+j+1}^+) - (a_1 - X_{n+j+1}^+),
$$

(29)

where $a_1 \equiv L - (x_{n+1} + \cdots + x_{n+j}) \leq a_2 \equiv L - (x_{n+1} + \cdots + x_{n+1})$. The inequality in (29) may be simplified by distinguishing among the following three cases:

$$
0 \leq X_{n+j+1} \leq a_1; \quad (29) \quad \Rightarrow \quad a_2 - a_1 \geq a_2 - a_1
$$

$$
X_{n+j+1} > a_1; \quad (29) \quad \Rightarrow \quad (a_2)^+ - (a_1)^+ \geq 0.
$$

The inequalities clearly hold in these cases.

(b) Note that $\zeta_j \equiv E \Delta J_{j,n} \leq P(X_{n+j} + \cdots + X_{n+1} \leq L) \int_{0}^{L} \beta(t) dt \leq (ML^{1/n})(1 + P(X_{n+j} + \cdots + X_{n+1} \leq L))$. The proof that the right-hand side is bounded by a geometrically decreasing sequence is analogous to that of (21).

(c) The expected backlogging costs charged to the nth demand epoch, i.e., the total expected backlogging costs incurred during $[S_n + L, S_{n+1} + L)$, are given by

$$
B(y_n) = E \sum_{j=0}^{\infty} \Delta J_{j,n} B_{S_{n+1}+L} = \sum_{j=0}^{\infty} E \Delta J_{j,n} E B_{S_{n+1}+L} = \sum_{j=0}^{\infty} \xi_j E B_{S_{n+1}+L}.
$$

(30)

The first equality follows from Fubini’s Theorem. To substantiate the second equality, note that $\Delta J_{j,n}$ is a function of $\{X_{n+1}+X_{n+2}, \ldots, X_{n+j}\}$ at the same time, by (19), $B_{S_{n+1}+L}$ is a function of $\{y_n, Z_{n+1}, Z_{n+2}, \ldots, Z_{n+1}\}$ and $y_n$ is a function of $\{X_1, X_2, \ldots, X_2, \ldots, Z_n\}$. By the assumptions of the compound renewal process, $\Delta J$ and $B_{S_{n+1}+L}$ are independent.
We next use the expression of \( B_{s+t}, \) in (19) to rewrite (30) as follows:

\[
B(y_n) = \sum_{j=1}^{\infty} \xi_j E \left[ B_{s+t}, \right] = \sum_{j=1}^{\infty} \xi_j E \left[ \sum_{i=1}^{j-1} \left( B_{s+t}, - \sum_{i=1}^{j-1} B_{s+t}, \right) \right] \\
= \sum_{j=1}^{\infty} (\xi_j - \xi_{j+1}) E \left[ \sum_{i=1}^{j} B_{s+t}, \right] \\
= \sum_{j=1}^{\infty} (\xi_j - \xi_{j+1}) E \left[ \sum_{i=1}^{j} (Z_{n+1} - y_n) \right].
\]

Thus the total expected backlogging costs charged to the nth demand epoch depend on the process history until that epoch only via the epoch’s inventory position after ordering, \( y_n \).

Thus, \( B(y_n) < \infty \) follows from part (b) and that for all \( j \geq 1, \ E[\sum_{i=1}^{j} (Z_{n+1} - y_n)] \leq \ E[\sum_{i=1}^{j} Z_{n+1}] = j \ E[Z] \). The convexity of \( B(\cdot) \) is immediate from (31) and because \( \xi_j \geq \xi_{j+1} \) by Lemma 4(a).

**Proof of Theorem 5:** Let II denote the class of all (possibly history dependent) policies. Let \( g(x) \) \( g(x) \) denote the long-run average cost under policy \( \pi \) in the original continuous review model [the periodic review model PR] when starting with an inventory position \( x \). It follows from Lemmas 3 and 4 that

\[
g_\pi(x) \geq \lim_{T \to \infty} \frac{1}{N(T)} \sum_{n=1}^{N(T)} E \left[ Kd(y_n - x_n) + c(y_n - x_n) + I(y_n) + B(y_n) \right] = \hat{g}_\pi(x)
\]

\[
g_\pi(x) = \hat{g}_\pi(x) \text{ whenever } \{ y_n | x_1 = x \text{ and policy } \pi \}
\]

is uniformly bounded from above.

Thus, \( \inf_{\pi \in \Pi} g_{\pi}(x) \geq \inf_{\pi \in \Pi} \hat{g}_{\pi}(x) \). Moreover, since \( B(\cdot) \) and \( I(\cdot) \) are finite and convex, and \( \lim_{t \to \infty} I(y) + B(y) = \infty \), it follows from Zheng (1991) that an \( (s, S) \) policy is optimal in the periodic review model PR. The unconventional feasibility constraint \( y_n \geq 0 \), implied by the (NIP) restriction, represents a slight variant of the standard model. However, by an argument similar to that in Theorem 2(a), one can show that an \( (s, S) \) policy with \( S^* \geq x^* \geq 0 \) is optimal under that nonnegativity constraint as well. Thus, by (32),

\[
\inf_{\pi \in \Pi} g_{\pi}(x) = \inf_{\pi \in \Pi} \hat{g}_{\pi}(x) = \hat{g}_{(s*, S*)} (x)
\]

\[
= g((s*, S*)) \geq \inf_{\pi \in \Pi} g_{\pi}(x),
\]

so that \( g_{(s*, S*)} (x) = \inf_{\pi \in \Pi} g_{\pi}(x) \) for all starting states \( x \). The second equality in (34) follows from (33) since, under the \( (s^*, S^*) \) policy, \( y_n \leq \max\{x, S^*\} \) almost surely.

**Endnote**

1. Assumption (NIP) fails to be applicable in the capacitated inventory system: if a period’s starting inventory position is below \(-C\), we are unable to increase the inventory position after ordering to a nonnegative level.

**References**


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