Modular Forms on a Function Field Over a Finite Field

by

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Modular Forms on a Function Field Over a Finite Field

Thesis directed by Professor Lynne Walling

This thesis extends the classical theory of modular forms over $\mathbb{Q}$ to the setting of a function field of a finite field. Using algebraic methods and harmonic analysis, especially the technique of Poisson summation, I am able to compute Fourier coefficients of Eisenstein series for both the full modular group and for congruence subgroups. For this setting let $\mathbb{F} = \mathbb{F}_q$ be a finite field of odd order $q = p^r$ and let T be an indeterminate. The ring of polynomials and its field of fractions are given by $\mathbb{A} = \mathbb{F}_q[T]$ and $\mathbb{K} = \mathbb{F}_q(T)$ with the valuation $\left|\cdot\right| = \left|\cdot\right|_\infty$ on $\mathbb{K}$ induced by the degree function as $\left|\frac{\alpha}{\beta}\right| = q^{\deg\alpha - \deg\beta}$ for $\alpha, \beta \in \mathbb{A}$. In the case of square-free level P, the β^{th} coefficient of Eisenstein series of (integer)weight k, square-free level N with character $\chi = \chi_1\chi_2$, and $D|N$ is

$$c(\beta,\tau) = Q(\beta,y)|D|^{1-k}|N| \sum_{m|\beta,\text{monic}} \chi_1(m)\chi_2\left(\frac{\beta}{m}\right)|m|^{1-k}$$

where $Q(\beta, y) = \left(\frac{1-q^{(d-2\ell)k}}{1-q^{-k}}\right)|y|^{2-k}|\beta|, \deg\beta = d$, and $\tau = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$. Similar result are found for the cases of level 1 and general level. Additionally, Hecke operators can be defined in certain situations and the action of these operators on the Eisenstein series is shown to be similar to the classical case.

Dedication

To my mother, Elisabeth, who always knew I could.

To Sami, Sharon, Harvey, Marc, Helene, Sara, Munira, Celia, Abram, and Salman who would have all been proud to have another doctor in the family.

To my partner, Barbara, your love helped see me through.

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Introduction

The theory of modular and automorphic forms developed around 1880 from a series of letters between F. Klein and H. Poincaré, some of which can be found in *Acta Math. 39* (1923). Later revolutionized in the 1930's by E. Hecke and his student H. Petersson, it has a rich history. These methods were prevalent in Ramanujan's work relating representation numbers to the tau function, $\tau(n)$, not surprising since it was Hardy who introduced the classical theta function, which is a modular form. The still open Lehmer conjecture claims $\tau(n) \neq 0$ for all $n \geq 1$. Petersson developed his famous inner product in 1938 which enabled the decomposition of each modular form into a linear combination of Eisenstein series and a cusp form, the bounding of Fourier coefficients of cusp forms, and most remarkably the simultaneous diagonalization of the Hecke operators. The field has continued to grow since that time.

Currently, researchers are using modular forms to construct Ramanujan and almost optimal expander graphs which have applications to communication networks. Modular forms are closely linked to elliptic curves, as in A. Wiles' solution to Fermat's last theorem and the subsequent work of Wiles and Taylor. They are useful tools in cryptography and coding theory, aiding in finding efficient algorithms to count points on curves and in constructing error-correcting codes. They generally appear in physics whenever a heat kernel occurs and with current work in random matrix theory may be very crucial in solving the problems of quantum computing. Many theorems about modular forms translate into results in seemingly different settings. This is seen, for example, in results relating certain lattices to Siegel modular forms. Modular forms have ties to many other areas such as random matrix theory, representation theory, Diophantine equations, and arithmetic geometry. In this way, the theory of modular and automorphic forms provides a bridge between analysis, algebra, and geometry.

Function fields, which are finite extensions of $k(T)$ for some field k, usually but not always taken to finite, and indeterminate T, have a long history going back to Dedekind and Kronecker, but took new importance once it was shown that the only types of fields which satisfy certain notions of absolute value are number fields and function fields. The former are finite extensions of $\mathbb{Q}$, linked to modular forms by the study of Hilbert and Hilbert-Siegel modular forms. Currently, much less is known about modular forms over function fields. Even the definitions of the "upper half-plane" or an exponential are not completely agreed upon. This is seen in the differences of Weil's and Drinfeld's half-planes, and Carlitz's exponential with the ones I will use. My approach is that of Hoffstein and Rosen as in [HR92] also used by Merrill and Walling in [MW93] and [HMW99]. This method also differs from the representation theoretic approach of Fisher and Freidberg in their recent papers.

My research involves extending the theory of modular forms in the setting of a function field over a finite field. Several analogies to the classical situation exist. The ring $\mathbb{A} = \mathbb{F}_q[T]$, usually $q = p^r$ odd, plays the role of $\mathbb{Z}$ with $SL_2(\mathbb{A})$ taking the place of the classical modular group $\Gamma = SL_2(\mathbb{Z})$. The expectation is that the theory of modular forms over a function field of a finite field should have some analogies to the classical theory over $\mathbb{Q}$.

Specifically, I have computed the β^{th} coefficient of the Fourier expansion of the integral weight Eisenstein series of polynomial level, N, with character χ, modulo N, with $\chi = \chi_1\chi_2$ and $D|N$. For

$$E_D(\tau) = \sum_{\substack{a \mod N \\ b \mod N/D \\ c \mod D}} \chi_1(b)\chi_2(c)e\left\{\frac{-ac}{D}\right\} G(\tau; bD, a; N),$$

the β^{th} coefficient is

$$c(\beta,\tau) = Q(\beta,y)|M|^{-k}|N|^2 \sum_{\substack{m' \in \mathbb{A}, \text{monic} \\ m'|\beta}} \chi_1(sm')\chi_2\left(\frac{\beta}{sm'}\right)|m'|^{1-k},$$

where $Q(\beta,y) = \left(\frac{1-q^{(d-2\ell)k}}{1-q^{-k}}\right)|y|^{2-k}|\beta|, \tau = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}, \quad y = T^{-2\ell}, M = (m,N), \deg\beta = d$, and $s = \frac{M}{D}$.

In the classical case, Γ is a discrete subgroup of $SL_2(\mathbb{R})$ which acts discontinuously on the complex upper half-plane, $\mathfrak{H} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$, by fractional linear transformation, $z \mapsto \frac{az+b}{cz+d}$. A modular form of weight k, level N, and character χ is an analytic function $f : \mathfrak{H} \to \mathbb{C}$ such that

(1)

$$f(\gamma z) = (cz+d)^k \chi(d) f(z)$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : N \mid c\right\}$

(2) $\lim_{z\to i\infty}(cz+d)^{-k}f(\gamma z) < \infty$ for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

The first condition gives f, our modular form, period 1, and hence a Fourier expansion $f(z) = \sum_{n\geq 0} a_n \exp(2\pi i n z)$. The coefficients a_n, in many cases, are known to encode a great deal of number theoretical information as in the case of the τ function; it is conjectured that this holds in all cases.

The images of ∞ under $\mathrm{SL}_2(\mathbb{Z})$ are called the **cusps** and condition (ii) is sometimes referred to as f **is holomorphic at the cusps.** If f vanishes at all the cusps (so in particular $a_0 = 0$), then f is called a **cusp form.** The space of cusp forms of weight k, level N, and character χ, denoted $\mathfrak{S}_k(N,\chi)$, is a subspace of $\mathfrak{M}_k(N,\chi)$, the space of modular forms of weight k, level N, and character χ. The **Eisenstein series**, defined as

$$E_k(\tau) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} (c_\gamma\tau + d_\gamma)^{-k} \qquad \text{where} \quad \gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix},$$

are of interest as the space spanned by the Eisenstein series is the orthogonal complement of the space of cusp forms. Classically, the Eisenstein series parts of a modular form are considered well understood so more focus has been on the cusp forms which due to bounds on the coefficients can be viewed as "error terms". However, over function fields of a finite field it is known that there are no nonzero cusp forms for the full modular group. Clearly the Eisenstein series will play a more dominant role in this setting.

In this function field case, starting with the ring $\mathbb{A} = \mathbb{F}_q[T]$ and field of fractions $\mathbb{K} = \mathbb{F}_q(T)$, the half-plane is defined as $SL_2(\mathbb{K}_\infty)/SL_2(\mathcal{O}_\infty)$ where $\mathbb{K}_\infty = \mathbb{F}_q((\frac{1}{T}))$ and $\mathcal{O}_\infty = \mathbb{F}_q[[\frac{1}{T}]]$ are the completion of $\mathbb{K}$ and ring of integers of $\mathbb{K}_\infty$ under the valuation $\left|\frac{\alpha}{\beta}\right| = q^{\deg\alpha - \deg\beta}$. This is not the same half-plane as Weil or Drinfeld, who used different settings, however this is a natural choice as $SL_2(\mathcal{O}_\infty)$ is a maximal compact subgroup of $SL_2(\mathbb{K}_\infty)$. Explicit sets of coset representatives were found (by Merrill and Walling in [MW93] and [HMW99]) for $\mathfrak{H}, \Gamma/\mathfrak{H}$, and $\Gamma_\infty/\mathfrak{H}$ via the Iwasawa decomposition, where

$$\Gamma_\infty = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c = 0 \right\} = \left\{ \begin{pmatrix} u & \alpha \\ 0 & u^{-1} \end{pmatrix} \in \Gamma : u \in \mathbb{F}^\times, \alpha \in \mathbb{A} \right\},$$

is the analog of the classical "translation subgroup."

Functions $f : \mathfrak{H} \longrightarrow \mathbb{C}$ which are invariant under the (left)action of Γ_∞ can be viewed as functions on the finite abelian (additive) subgroup $T^{1+2m}\mathbb{A}/\mathbb{A}$ of $\mathbb{K}_\infty/\mathbb{A}$. This is accomplished by considering $f_y(x) = f\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right)$ for fixed $y = T^{2m}$. Naturally, Fourier series in x becomes a valuable in analyzing these functions. For $x \in \mathbb{K}_\infty$,write $x = \sum_{j=-\infty}^{N} x_j T^j$, and let $e\{x\} = \exp\left(\frac{2\pi i \operatorname{Tr}(x_1)}{p}\right)$, where the trace is from $\mathbb{F}$ to $\mathbb{Z}/p\mathbb{Z}$. Basic harmonic analysis yields a Fourier expansion for $f_y(x) \Leftrightarrow \sum_{\beta \in T^2\mathbb{A}} c_\beta(f, y) e\{\beta x\}$. I am interested in these coefficients since, as Thakur recently wrote, "in general, the coefficients of q_∞-expansions of modular forms, which as very rich arithmetically in the classical case, are very poorly understood objects so far." [Tha04]

Chapter 2

Function Field Preliminaries

This chapter will describe the function field setting: defining our primary objects, giving sets of coset representatives for a halfplane and a fundamental domain, and applying Fourier analysis with an appropriate exponential. The definitions in section 2.1 are conveniently listed at the end in table 2.1 with their analogies to the classical case. The treatment is similar to those in [HR92], [MW93], and [HMW99].

2.1 Definitions

Let $\mathbb{F} = \mathbb{F}_q$ be a finite field of odd order $q = p^r$ and let T be an indeterminate. The ring of polynomials and its field of fractions are given by $\mathbb{A} = \mathbb{F}_q[T]$ and $\mathbb{K} = \mathbb{F}_q(T)$. Defining the valuation $\left|\cdot\right| = \left|\cdot\right|_\infty$ on $\mathbb{K}$ induced by the degree function as

$$\left|\frac{\alpha}{\beta}\right| = q^{\deg\alpha - \deg\beta} \text{ for } \alpha, \beta \in \mathbb{A}$$

and agreeing that $\left|0\right| = -\infty$, the completion of $\mathbb{K}$ with respect to this valuation is $\mathbb{K}_\infty = \mathbb{F}_q((\frac{1}{T}))$. Extend the degree function in the natural way to $\mathbb{K}_\infty$ with $\deg y = \operatorname{ord}_{\frac{1}{T}} y$. Therefore the ring of integers for the completed field, $\mathcal{O}_\infty = \left\{\alpha \in \mathbb{K}_\infty : \left|\alpha\right| \le 1\right\}$, is given by $\mathbb{F}_q[[\frac{1}{T}]]$. It is often convenient to view $\mathbb{K}_\infty$ as $\left\{\sum_{n=-\infty}^{N} \alpha_n T^n : \alpha_n \in \mathbb{F}\right\}$ and $\mathcal{O}_\infty$ as $\left\{\sum_{n=-\infty}^{N} \alpha_n T^n : \alpha_n \in \mathbb{F}, N \ge 0\right\}$ i.e. the formal Laurent series and Taylor series respectively in $\frac{1}{T}$. Notably $\frac{1}{T}$ is a local uniforming parameter and $\mathcal{O}_\infty$ is a discrete valuation ring with a unique maximal ideal $\frac{1}{T}\mathcal{O}_\infty = \mathcal{P}_\infty$. It is easily seen that $\mathbb{K}_\infty =$

$\mathbb{A} + \mathcal{O}_\infty$, $\mathbb{A} \bigcap \mathcal{O}_\infty = \mathbb{F}$, $\mathbb{A}$ is a discrete subgroup of $\mathbb{K}_\infty$, and $\mathbb{K}_\infty/\mathbb{A}$ is compact. There is an obvious isomorphism between $\mathbb{K}_\infty/\mathbb{A}$ and $\mathcal{P}_\infty$.

In the classical case the "upper half-plane" is defined to be $SL_2(\mathbb{R})$ modulo the maximal compact subgroup $SO_2(\mathbb{R})$. Noting that the group $\mathrm{PSL}_2(\mathcal{O}_\infty)$ is the maximal compact subgroup of $\mathrm{PSL}_2(\mathbb{K}_\infty)$ an "upper half-plane" $\mathfrak{H}$ may be defined as

$$\mathfrak{H} = \mathrm{PSL}_2(\mathbb{K}_\infty) \diagup \mathrm{PSL}_2(\mathcal{O}_\infty).$$

Of particular note is the discrete subgroup $\Gamma = \mathrm{SL}_2(\mathbb{A})$ and its subgroup

$$\Gamma_\infty = \left\{ \begin{pmatrix} u & \alpha \\ 0 & u^{-1} \end{pmatrix} : u \in \mathbb{F}^\times, \alpha \in \mathbb{A} \right\}.$$

At this point the analogs to the classical case are clear. There is a ring $\mathbb{A}$ or $\mathbb{Z}$, a field of fractions $\mathbb{K}$ or $\mathbb{Q}$, a completion $\mathbb{K}_\infty$ or $\mathbb{R}$ (or $\mathbb{Q}_p$) , and its ring of integers $\mathcal{O}_\infty$ or $\mathbb{Z}$ (or $\mathbb{Z}_p$). The following lemma (cf.[HMW99]) gives a description of a complete set of coset representatives of $\mathfrak{H}$, the "upper half-plane" .

Lemma 2.1.1 (Iwasawa decomposition) The set

$$\left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} : y = T^{2m}, m \in \mathbb{Z}, x \in T^{2m+1}\mathbb{A} \right\}$$

is a complete set of coset representatives for $\mathfrak{H} = \mathrm{PSL}_2(\mathbb{K}_\infty) \diagup \mathrm{PSL}_2(\mathcal{O}_\infty)$.

Proof: Take $z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{K}_\infty)$. Then, depending on the degrees of c and d either $\begin{pmatrix} 1 & 0 \\ -\frac{c}{d} & 1 \end{pmatrix}$ or $\begin{pmatrix} \frac{d}{c} & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{PSL}_2(\mathcal{O}_\infty)$. In either case,

$$z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} w & x' \\ 0 & w^{-1} \end{pmatrix} \text{ for } w, x' \in \mathbb{K}_\infty, w \neq 0.$$

Writing $w = T^m u$ for $u \in \mathcal{O}_\infty^\times$ and $m \in \mathbb{Z}$, observe that

$$z \equiv \begin{pmatrix} w & x' \\ 0 & w^{-1} \end{pmatrix} \begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix} \equiv \begin{pmatrix} T^m & x'u \\ 0 & T^{-m} \end{pmatrix}.$$

However $x'u = T^{-m}(x + T^{2m}v)$ for $x \in T^{2m+1}\mathbb{A}$ and $v \in \mathcal{O}_\infty$, so

$$z \equiv \begin{pmatrix} T^m & x'u \\ 0 & T^{-m} \end{pmatrix} \begin{pmatrix} 1 & -v \\ 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} T^m & xT^{-m} \\ 0 & T^{-m} \end{pmatrix}.$$

To see that this is unique, suppose

$$z \equiv \begin{pmatrix} T^m & xT^{-m} \\ 0 & T^{-m} \end{pmatrix} \equiv \begin{pmatrix} T^{m'} & x'T^{-m'} \\ 0 & T^{-m'} \end{pmatrix},$$

then

$$\begin{pmatrix} T^m & xT^{-m} \\ 0 & T^{-m} \end{pmatrix} \begin{pmatrix} T^{m'} & x'T^{-m'} \\ 0 & T^{-m'} \end{pmatrix} = \begin{pmatrix} T^{m'-m} & (x'-x)T^{-m'-m} \\ 0 & T^{m-m'} \end{pmatrix} \in \mathrm{PSL}_2(\mathcal{O}_\infty).$$

Thus $T^{m'-m}$, $T^{m-m'} \in \mathcal{O}_\infty$ so $m = m'$. Considering that $(x - x')T^{-2m} \in \mathcal{O}_\infty$ and additionally in $T\mathbb{A}$ it follows that $x = x'$ and the representation is unique. □

The modular group $\Gamma = \mathrm{SL}_2(\mathbb{A})$ acts on $\mathfrak{H}$ by left multiplication, with subgroup Γ_∞ being the stabilizer of the cusp at ∞. The next two lemmata from [MW93] and [HMW99] give coset representatives for $\Gamma_\infty \diagdown \mathfrak{H}$ with

$$\Gamma_\infty = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c = 0 \right\} = \left\{ \begin{pmatrix} u & \alpha \\ 0 & u^{-1} \end{pmatrix} \in \Gamma : u \in \mathbb{F}^\times, \alpha \in \mathbb{A} \right\},$$

an analog of the "translation group" or "vertical strip" , and fundamental domain, $\Gamma \diagdown \mathfrak{H}$.

Lemma 2.1.2 The set

$$\left\{ \begin{pmatrix} T^{2m} & 0 \\ 0 & 1 \end{pmatrix} : m \geq 0 \right\} \bigcup \left\{ \begin{pmatrix} T^{2m} & x \\ 0 & 1 \end{pmatrix} : m \leq 0, x \in T^{2m+1}\mathbb{A} \cap \frac{1}{T}\mathcal{O}_\infty \right\}$$

is a complete set of coset representatives for $\Gamma_\infty \diagdown \mathfrak{H}$ with x and y as in Lemma 2.1.3

Proof: Take $\tau \in \mathfrak{H}$ so $\tau = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$ with x and y as in Lemma 2.1.3. So $y = T^{2m}$ for $m \in \mathbb{Z}$, and $x \in T^{2m+1}\mathbb{A}$. The group action insures that $\tau\gamma \in \mathfrak{H}$ for all

$$\gamma \in \Gamma_\infty = \left\{ \begin{pmatrix} u & \alpha \\ 0 & u^{-1} \end{pmatrix} \in \Gamma : u \in \mathbb{F}^\times, \alpha \in \mathbb{A} \right\}.$$

$$\tau\gamma = \begin{pmatrix} yT^{2m} & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u & \alpha \\ 0 & u^{-1} \end{pmatrix} = \begin{pmatrix} uT^{2m} & \alpha T^{2m} + u^{-1}x \\ 0 & u^{-1} \end{pmatrix}$$

The lemma follows by observing when $\alpha T^{2m} + u^{-1}x \in T^{2m+1}\mathbb{A}$.

See A. Weil "On the analogue of the modular group in characteristic p" □

Lemma 2.1.3 (Fundamental domain) The set

$$\left\{ \begin{pmatrix} T^{2m} & 0 \\ 0 & 1 \end{pmatrix} : m \geq 0 \right\}$$

is a complete set of coset representatives for $\Gamma \diagdown \mathfrak{H}$.

Proof: The proof follows from two observations. First, that $\Gamma = \mathrm{SL}_2(\mathbb{A})$ is generated by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\left\{ \begin{pmatrix} u & a \\ 0 & u^{-1} \end{pmatrix} : a \in \mathbb{A}, u \in \mathbb{F}^\times \right\}$. Secondly, that for $\tau = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \in \mathfrak{H}$ with $x \in T^{2m+1}\mathbb{A}$,

$$-\frac{1}{\tau} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tau \equiv \begin{cases} \begin{pmatrix} T^{2m} & 0 \\ 0 & 1 \end{pmatrix} & \text{if } x = 0 \\ \begin{pmatrix} T^{n-2m} & -(\frac{1}{x})T^{2m-n} \\ 0 & T^{2m-n} \end{pmatrix} & \text{if } x \neq 0, \deg x = -n. \end{cases}$$

□

These results are summarized in the following table.

Classical		Function Field
$\mathbb{Z}$	**Ring**	$\mathbb{A} = \mathbb{F}_q[T]$
$\mathbb{Q}$	**Fraction field**	$\mathbb{K} = \mathbb{F}_q(T)$
$\lvert \cdot \rvert$ or $\lvert \cdot \rvert_p$	**Valuation**	$\lvert \frac{\alpha}{\beta} \rvert = q^{\deg \alpha - \deg \beta}$
$\mathbb{R}$ or $\mathbb{Q}_p$	**Completion**	$\mathbb{K}_\infty = \mathbb{F}_q((\frac{1}{T}))$ $\left\{ \sum_{n=-\infty}^{N} \alpha_n T^n \,:\, \alpha_n \in \mathbb{F}_q \right\}$
$\mathbb{Z}$ or $\mathbb{Z}_p$	**Ring of integers**	$\mathcal{O}_\infty = \mathbb{F}_q[[\frac{1}{T}]]$ $\left\{ \sum_{n=-\infty}^{N} \alpha_n T^n \,:\, \alpha_n \in \mathbb{F}_q, N \leq 0 \right\}$
$\mathrm{SL}_2(\mathbb{R}) / \mathrm{SO}_2(\mathbb{R})$	**Upper half-plane**	$\mathrm{PSL}_2(\mathbb{K}_\infty) / \mathrm{PSL}_2(\mathcal{O}_\infty)$ $\left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} : \begin{smallmatrix} y = T^{2m}, m \in \mathbb{Z}, \\ x \in T^{2m+1}\mathbb{A} \end{smallmatrix} \right\}$
$\Gamma = SL_2(\mathbb{Z})$	**Modular group**	$\Gamma = SL_2(\mathbb{A})$
Γ_∞ $\left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \Gamma \right\}$	**Translation subgroup**	Γ_∞ $\left\{ \begin{pmatrix} u & \alpha \\ 0 & u^{-1} \end{pmatrix} \in \Gamma : \alpha \in \mathbb{A}, u \in \mathbb{F}^\times \right\}$
$\Gamma_\infty \backslash \mathfrak{H}$ $\left\{ z \in \mathbb{C} : \begin{smallmatrix} -\frac{1}{2} \leq \mathrm{Re}(z) < \frac{1}{2} \\ \mathrm{Im}(z) > 0 \end{smallmatrix} \right\}$	**"Vertical strip"**	$\Gamma_\infty \backslash \mathfrak{H}$ $\left\{ \begin{pmatrix} T^{2m} & 0 \\ 0 & 1 \end{pmatrix} : m \in \mathbb{Z}, m \geq 0 \right\} \bigcup$ $\left\{ \begin{pmatrix} T^{2m} & x \\ 0 & 1 \end{pmatrix} : \begin{smallmatrix} m \in \mathbb{Z}, m < 0, \\ x \in T^{2m+1}\mathbb{A} \cap \frac{1}{T}\mathcal{O}_\infty \end{smallmatrix} \right\}$
$\Gamma \backslash \mathfrak{H}$ $\left\{ z \in \mathbb{C} : \begin{smallmatrix} -\frac{1}{2} \leq \mathrm{Re}(z) < \frac{1}{2}, \\ \mathrm{Im}(z) > 0, \lvert z \rvert \geq 1, \\ \text{if } \lvert z \rvert = 1 \\ \text{then } \mathrm{Re}(z) \geq 0 \end{smallmatrix} \right\}$	**Fundamental domain**	$\Gamma \backslash \mathfrak{H}$ $\left\{ \begin{pmatrix} T^{2m} & 0 \\ 0 & 1 \end{pmatrix} : m \in \mathbb{Z}, m \geq 0 \right\}$

Table 2.1: Dictionary

2.2 Fourier Series

As in the classical case the modular group, $\Gamma = SL_2(\mathbb{A})$, acts by left multiplication on the fundamental domain, $\Gamma \backslash \mathfrak{H}$. It is expected that some functions $f : \mathfrak{H} \longrightarrow \mathbb{C}$ will transform under this group action. In fact, these functions should be invariant under the action of the stabilizer of the cusps (in this case just the cusp at ∞), Γ_∞.

For such a function f, fix $y = T^{-2\ell}$ and set the function on $T^{1-2\ell}\mathbb{A}$ to be

$$f_y(x) = f\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right).$$

Due to its invariance under Γ_∞, f_y can be viewed as a function on the finite abelian subgroup $T^{1-2\ell}\mathbb{A}/\mathbb{A}$ of $\mathbb{K}_\infty/\mathbb{A}$ making Fourier series in x a valuable tool. First define the exponential.

Definition 2.2.1 For $x \in \mathbb{K}_\infty$,write $x = \sum_{j=-\infty}^{N} x_j T^j$, and define the **exponential** $e\{x\}$ by

$$e\{x\} = \exp\left(\frac{2\pi i \operatorname{Tr}(x_1)}{p}\right)$$

where the trace is from $\mathbb{F}_q$ to $\mathbb{Z}/p\mathbb{Z}$.

Additionally, choose μ to be an (additive) left Haar measure on $\mathbb{K}_\infty$, normalized so that $\mu(\mathcal{O}_\infty) = 1$. All the following integrals written dx will be with respect to μ, its restrictions or projections.

Elementary Fourier analysis and the following Theorem from [MW93] then shows that any f invariant under the action of Γ_∞ can be written as a Fourier series

$$\sum_{\beta \in T^2\mathbb{A}} I_{\mathcal{O}_\infty}(\beta y) c_\beta(f, y) e\{\beta x\}$$

where $I_{\mathcal{O}_\infty}$ is the characteristic function of $\mathcal{O}_\infty$.

Theorem 2.2.2 (1) The character groups of $\mathbb{K}_\infty$, $\mathbb{K}_\infty/\mathbb{A}$, and $T^{1-2m}\mathbb{A}/\mathbb{A}$ are isomorphic to $\mathbb{K}_\infty$, $T^2\mathbb{A}$, and $T^2\mathbb{A}/T^{2m+1}\mathbb{A}$ respectively.

(2) Any function f on $\mathfrak{H}$ which is invariant under the action of Γ_∞ can be expanded in a Fourier series

$$f\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = \sum_{\beta \in T^2\mathbb{A}} I_{\mathcal{O}_\infty}(\beta y) c_\beta(f,y) e\{\beta x\},$$

where $I_{\mathcal{O}_\infty}$ is the characteristic function of $\mathcal{O}_\infty$ and

$$c_\beta(f,y) = q^{1-2\ell} \sum_{x \in T^{1-2\ell}\mathbb{A}/\mathbb{A}} f_y(x) e\{-\beta x\}.$$

Paralleling the classical case, the computation of the Fourier coefficient, $c_\beta(f,y)$, depends on the type of function f.

If $f \in L^1(\mathbb{K}_\infty)$ then the coefficient is

$$c_\beta(f,y) = \int_{\mathbb{K}_\infty} f_y(x) e\{-\beta x\} dx,$$

if $f \in L^1(\mathbb{K}_\infty/\mathbb{A})$ then the coefficient is

$$c_\beta(f,y) = q \int_{\mathbb{K}_\infty/\mathbb{A}} f_y(x) e\{-\beta x\} dx,$$

and if $f \in L^1(T^{1-2\ell}\mathbb{A}/\mathbb{A})$ then the coefficient is

$$c_\beta(f,y) = q^{1-2\ell} \sum_{x \in T^{1-2\ell}\mathbb{A}/\mathbb{A}} f_y(x) e\{-\beta x\}.$$

Observing that $f_y(x) = \sum_{\beta \in \mathrm{T}^2\mathbb{A}} c_\beta(f,y) e\{\beta x\}$ where

$$c_\beta(f,y) = I_{\mathcal{O}_\infty}(\beta y)\, q^{1+2m} \sum_{x \in T^{2m+1}\mathbb{A}/\mathbb{A}} f_y(x) e\{-\beta y\},$$

it is evident that $\beta y \notin \mathcal{O}_\infty$ implies $c_\beta(f,y) = 0$. Consequently, $f_y(x) = c_0(f,y)$ whenever $\deg y \geq 0$. The following proposition (cf.[HMW99]) results.

Proposition 2.2.3 There are no nonzero cusp forms.

Proof: Suppose f is a cusp form. Then for any $z = \begin{pmatrix} T^{2m} & 0 \\ 0 & 1 \end{pmatrix}$ in the fundamental domain, $\Gamma \diagdown \mathfrak{H}$, $m \geq 0$. Thus $f_y(x) = c_0(f,y) = 0$. As f is zero on the fundamental domain it is identically zero. $\square$

2.3 Hecke Operators

In keeping with classical analogs, we define Hecke operators, however in this situation this is only possible for irreducible monic polynomials of **even** degree. The degree must be even for $\begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}$ to preserve the upper half-plane. With $\Gamma = \mathrm{SL}_2(\mathbb{A})$ and for P an irreducible monic of even degree in $\mathbb{A}$, let $\delta = \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}$ and set $\Gamma' = \delta\Gamma\delta^{-1}$. We may define for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{K}_\infty)$ the "slash" operator

$$f \left| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right. (\tau) = |ad - bc|^{\frac{k}{2}} |c\tau + d|^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right)$$

The following lemma gives a complete set of coset representatives for $\Gamma \cap \Gamma' \backslash \Gamma$.

Lemma 2.3.1 Given $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{cases} \begin{pmatrix} 1 & b' \\ 0 & 1 \end{pmatrix} & \text{if } P \nmid a \\ \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix} & \text{if } P | a \end{cases}$$

where $b \equiv ab' \mod P$ and the equivalence is by cosets of $\Gamma \cap \Gamma' \backslash \Gamma$.

Proof: Suppose $P \nmid a$. Noting that $\Gamma \cap \Gamma' \subseteq \begin{pmatrix} \mathbb{A} & P\mathbb{A} \\ \mathbb{A} & \mathbb{A} \end{pmatrix}$. The coset $(\Gamma \cap \Gamma') \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -b' \\ 0 & 1 \end{pmatrix} = (\Gamma \cap \Gamma') \begin{pmatrix} a & b - ab' \\ c & d - cb' \end{pmatrix}$. Since $b - ab' \in P\mathbb{A}$, $b \equiv ab' \mod P$ and as b runs modulo P the cosets are distinct.

Now in the case that $P|a$, look at $\gamma' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \frac{1}{P} & 0 \\ 0 & P \end{pmatrix} = \begin{pmatrix} \frac{a}{P} & bP \\ \frac{c}{P} & dP \end{pmatrix} \subseteq \Gamma'$. So

$$\delta^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \delta^{-1}\gamma' \begin{pmatrix} P & 0 \\ 0 & \frac{1}{P} \end{pmatrix} = \gamma\delta^{-1} \begin{pmatrix} P & 0 \\ 0 & \frac{1}{P} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{P} \end{pmatrix} = \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}$$

where equality is of cosets in $\Gamma \cap \Gamma' \diagdown \Gamma$. □

Chapter 3

Eisenstein Series

In the classical case, Eisenstein series are considered "fully understood" so the cusp forms, which due to the bounds on Fourier coefficients essentially are error terms, became the focus of researchers' attention. However, as shown in section 2.2 there are no nonzero cusp forms of the full modular group for function fields of a finite field. So in this setting researching the role of the Eisenstein series dominates.

3.1 Level 1

When the level of a modular form f is 1 (or any unit in $\mathbb{A}$, which are elements of $\mathbb{F}^{\times}$) , we say f transforms under the full modular group Γ.

Definition 3.1.1 The **Eisenstein series of (integer)weight k, level 1** is defined to be

$$E_k(\tau) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} |c\tau + d|^{-k}$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\tau = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$.

In the function field case a Zeta function can be defined in a closed form as

$$\zeta(s) = \sum_{\substack{g \in \mathbb{A} \\ g \text{ monic}}} |g|^{-s} = \sum_{n \geq 0} \frac{q^n}{q^{ns}} = \frac{1}{1 - q^{1-s}}.$$

Now define

$$A_k(\tau) = \zeta(k)(E_k(\tau) - (-1)^k) = \sum_{\substack{m,\, n \,\in\, \mathbb{A} \\ m \text{ monic}}} |(my, mx+n)|^{-k}$$

and take the Fourier expansion of $A_k(\tau)$, by

$$A_{k,y}(x) = \sum_{\beta \in \mathrm{T}^2\mathbb{A}} c_\beta(y,k) e\{\beta x\}.$$

The main result of this section is to explicitly determine this Fourier coefficient $c_\beta(y,k)$. First a lemma.

Lemma 3.1.2 For $\beta \in T^2\mathbb{A}, \beta \neq 0$, let $-n = \deg \beta y \leq 0$. Then

$$\int_{\mathbb{K}_\infty} \frac{e\{-\beta x\}}{|(y,x)|^k} dx = \frac{|y|^{1-k}}{1-q^{1-k}} \left[(1-q^{-k})(1-q^{(n+1)(1-k)}) \right]. \tag{3.1}$$

Proof: By the change of variables $x \mapsto xy$, and since

$$\int_{\mathcal{O}_\infty} \frac{e\{-\beta x_0 y\}}{|(1, x+x_0)|^k} dx_0 = |(1,x)|^{-k}$$

for any $x \in \mathbb{K}_\infty$, when writting $x = x_0 + v$ for $x_0 \in \mathcal{O}_\infty$ and $v \in \mathbb{K}_\infty / \mathcal{O}_\infty \approx T\mathbb{A}$,

$$\int_{\mathbb{K}_\infty} \frac{e\{-\beta x\}}{|(y,x)|^k} dx = |y|^{1-k} \int_{\mathbb{K}_\infty} \frac{e\{-\beta xy\}}{|(1,x)|^k} dx = |y|^{1-k} \sum_{x \in\, \mathbb{K}_\infty/\mathcal{O}_\infty} \frac{e\{-\beta xy\}}{|(1,x)|^k}.$$

This sum can be evaluated by looking at cases for fixed $\deg x = D$.

Clearly the case with $x = 0$ contributes $|y|^{1-k}$.

The next case is when $n \geq D$. Here $x\beta y \in \mathcal{O}_\infty$ and $e\{-x\beta y\} = 1$, so this is a matter of counting.

Third is the case with $n = D-1$. Begin by writing $x = \sum_{i=1}^{D} c_i T^i$ and $-\beta y = \sum_j b_j T^j$, then look at the coefficient of T in $-x\beta y$. As $-x\beta y = \sum_\ell (\sum_{i+j=\ell} c_i b_j) T^\ell$ this coefficient is

$$\sum_{i+j=1} c_i b_j = \sum_{\substack{j \,\leq\, 1-D \\ 1 \,\leq\, i \,\leq\, D \\ i+j \,=\, 1}} c_i b_j = c_D b_{1-D}.$$

Since both $c_D \neq 0$ and $b_{1-D} \neq 0$, $c_D b_{1-D}$ varies over $\mathbb{F}^\times$ as c_D does. This makes

$$\sum_{\substack{x \in T\mathbb{A} \\ \deg x = D}} e\{-x\beta y\} = \sum_{\substack{u \in \mathbb{F}^\times \\ x = uT^D + x'}} \exp\left(2\pi i \operatorname{Tr}\left(u b_{1-D}\right)/p\right)$$
$$= \{\#\text{ of } x'\text{ s}\} \cdot \sum_{u \in \mathbb{F}^\times} \exp\left(2\pi i \operatorname{Tr}\left(u b_{1-D}\right)/p\right).$$

The sum is an incomplete character sum and equal to -1, while the $\{\#\text{ of } x'\text{ s}\} = q^{D-1}$ hence the contribution is $|y|^{1-k}\left(-q^{n-kD}\right)$.

Finally the case with $n < D-1$. Take x and $-\beta y$ as above but now $D \geq n+2$ so the relevant term in x of $\sum_{\ell}(\sum_{i+j=\ell} c_i b_j)T^\ell$ when ℓ is 1 is no longer c_D but another c_i which varies over all of $\mathbb{F}$ instead of $\mathbb{F}^\times$. So

$$\sum_{\substack{x \in T\mathbb{A} \\ \deg x = D}} e\{-x\beta y\} = \sum_{u \in \mathbb{F}^\times} \sum_{\substack{x' \in T\mathbb{A} \\ \deg x' \leq D}} e\{-\beta y\left(x' + uT^D\right)\}$$
$$= \sum_{u \in \mathbb{F}^\times} e\{-\beta y u T^D\} \sum_{x' \in T\mathbb{A}/T\mathbb{A} \cap T^D \mathcal{O}_\infty} e\{-x'\beta y\}.$$

The last sum on u is a nontrivial character sum so it's contribution is zero. Combining the cases

$$\int_{\mathbb{K}_\infty} \frac{e\{-\beta x\}}{|(y,x)|^k} dx = |y|^{1-k}\left[1 - q^{n-kD} + (q-1)\, q^{-1} \sum_{1 \leq D \leq n} q^{D(1-k)}\right]$$
$$= \frac{|y|^{1-k}}{1-q^{1-k}}\left[\left(1-q^{1-k}\right)\left(1+q^{n(1-k)-k}\right) + q^{-k}(q-1)\left(1-q^{n-nk}\right)\right]$$
$$= \frac{|y|^{1-k}}{1-q^{1-k}}\left[\left(1-q^{-k}\right)\left(1-q^{(n+1)(1-k)}\right)\right]$$

□

With this lemma in hand the main result of this section follows. The coefficients of the Fourier expansion are explicitly computed.

Theorem 3.1.3 For the shifted Eisenstein series

$$A_k(\tau) = \zeta(k)(E_k(\tau) - (-1)^k) = \sum_{\substack{c,\, d \,\in\, \mathbb{A} \\ c \text{ monic}}} |(cy, cx+d)|^{-k}$$

with Fourier series in x given by

$$A_{k,y}(x) = \sum_{\beta \in \mathrm{T}^2\mathbb{A}} c_\beta(y,k) e\{\beta x\}$$

then the coefficient $c_\beta(y,k)$ is

$$c_\beta(y,k) = \sigma_{1-k}\left(\beta T^{-2}\right) |y|^{1-k} \left(\frac{\left(1-q^{-k}\right)\left(1-q^{(n+1)(1-k)}\right)}{1-q^{1-k}} \right). \tag{3.2}$$

Proof: The β^{th} Fourier coefficient of $A_{k,y}(x) = \sum_{\beta \in \mathrm{T}^2\mathbb{A}} c_\beta(y,k) e\{\beta x\}$ is

$$\begin{aligned}
c_\beta(y,k) &= q \int_{\mathbb{K}_\infty/\mathbb{A}} A_k(\tau) e\{-\beta x\} dx \\
&= q \int_{\mathbb{K}_\infty/\mathbb{A}} \sum_{\substack{c,\, d \,\in\, \mathbb{A} \\ c \,\neq\, 0, \text{ monic}}} \frac{e\{-\beta x\}}{|(cy, cx+d)|^k} dx && (\text{let } d = c\ell + r) \\
&= q \int_{\mathbb{K}_\infty/\mathbb{A}} \sum_{\substack{c,\, \ell \,\in\, \mathbb{A} \\ c \text{ monic} \\ r \mod c}} \frac{e\{-\beta x\}}{|(cy, c(x+\ell)+r)|^k} dx && (\text{let } x \mapsto x - \ell) \\
&= q \sum_{\substack{c \,\in\, \mathbb{A} \\ c \text{ monic}}} |c|^{-k} \sum_{r \mod c} \int_{\mathbb{K}_\infty} \frac{e\{-\beta x\}}{|(y, x+\frac{r}{c})|^k} dx && \left(\text{let } x \mapsto x - \frac{r}{c}\right) \\
&= q \sum_{\substack{c \,\in\, \mathbb{A} \\ c \text{ monic}}} |c|^{-k} \sum_{r \mod c} e\left\{\frac{\beta r}{c}\right\} \int_{\mathbb{K}_\infty} \frac{e\{-\beta x\}}{|(y,x)|^k} dx.
\end{aligned}$$

The Theorem follows by recognizing that

$$q \sum_{\substack{c \,\in\, \mathbb{A} \\ c \text{ monic}}} |c|^{-k} \sum_{r \mod c} e\left\{\frac{\beta r}{c}\right\} = \sigma_{1-k}\left(\beta T^{-2}\right)$$

and applying Lemma 3.1.2. □

3.2 Irreducible Level P

Before moving on to Eisenstein series of level N we must first define the congruence subgroups.

Definition 3.2.1 For $\Gamma = \mathrm{SL}_2(\mathbb{A})$ and $N \in \mathbb{A}$ **the principal congruence subgroup of level N** is

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : ad - bc \equiv 1 \mod N \right\}$$

Definition 3.2.2 For $\Gamma = \mathrm{SL}_2(\mathbb{A})$ and $N \in \mathbb{A}$ **the level N stabilizer of the cusps** is

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N) : N|c \right\}$$

For level $P \in \mathbb{A}$ the computation follows as in [Ogg69]. Begin by defining the G-Eisenstein series of level P.

Definition 3.2.3 The **G-Eisenstein series of (integer)weight k, irreducible level P** is defined to be

$$G(\tau; c, d; P) = \sum_{\substack{m, n \in \mathbb{A}, m \neq 0 \\ m \equiv c \mod P \\ n \equiv d \mod P}} |m\tau + n|^{-k}$$

where $\tau = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$.

Definition 3.2.4 The **Eisenstein series of (integer)weight k, irreducible level P with character χ modulo P** is

$$E_P(\tau) = \sum_{\substack{a \mod P \\ c \mod P}} \chi(c) e\left\{ \frac{-ac}{P} \right\} G(\tau; P, a; P).$$

First a technical lemma.

Lemma 3.2.5 With $y = T^{-2\ell}$ and $\deg \beta = d \leq 2\ell$

$$\int_{\mathcal{P}_\infty} \frac{e\{-\beta x\}}{|(y,x)|^k} dx = (q-1)Q(\beta, y)$$

where $Q(\beta, y) = \left(\frac{1-q^{(d-2\ell)k}}{1-q^{-k}}\right) |y|^{2-k}|\beta|$.

Proof:

$$\int_{\mathcal{P}_\infty} \frac{e\{-\beta x\}}{|(y,x)|^k} dx = |y|^{1-k} \int_{\mathcal{P}_\infty} \frac{e\{-\beta xy\}}{|(1,x)|^k} dx$$

$$= |y|^{1-k} \sum_{x_0 \in \mathcal{P}_\infty / \mathcal{P}_\infty^{2\ell-d}} \int_{\mathcal{P}_\infty^{2\ell-n}} \frac{e\{-\beta y(x_0+x)\}}{|(1,x_0+x)|^k} dx.$$

Noting that $|(1, x_0 + x)| = |(1, x_0)|$ and using Lemma 3.6 of [MW93] we now have

$$\int_{\mathcal{P}_\infty} \frac{e\{-\beta x\}}{|(y,x)|^k} dx = |y|^{1-k}|\beta y| \sum_{x_0 \in \mathcal{P}_\infty / \mathcal{P}_\infty^{2\ell-D}} \frac{e\{-\beta y x_0\}}{|(1,x_0)|^k}$$

$$= (q-1)\left(\frac{1-q^{(d-2\ell)k}}{1-q^{-k}}\right)|y|^{2-k}|\beta|$$

$\square$

The resulting theorem gives the β^{th} coefficient of the Fourier expansion of the G-Eisenstein series of (integer)weight k, level P in the above definition.

Proposition 3.2.6 The β^{th} coefficient of the Fourier expansion of G-Eisenstein series of (integer)weight k, irreducible level P,

$$G(\tau; P, a; P) = \sum_{\substack{m, n \in \mathbb{A}, m \neq 0 \\ (m,n) \equiv (0,a) \mod P}} |m\tau + n|^{-k} \quad \text{is}$$

$$c_\beta(y) = (q-1)Q(\beta, y)|P|^{-k} \sum_{\substack{m' \neq 0 \\ m' | \beta}} |m'|^{1-k} e\left\{\frac{\beta a}{m' P}\right\}$$

where $\tau = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$, $\quad y = T^{-2\ell}, \deg \beta = d$, and $Q(\beta, y) = \left(\frac{1-q^{(d-2\ell)k}}{1-q^{-k}}\right)|y|^{2-k}|\beta|$

Proof: The β^{th} coefficient of $G(\tau; P, a; P)$ is given by

$$\int_{\mathcal{P}_\infty} \sum_{\substack{m,n\in\mathbb{A} \\ m\neq 0 \\ (m,n)\equiv(0,a) \mod P}} e\{-\beta x\}|(my, mx+n)|^{-k}dx =$$

$$\int_{\mathcal{P}_\infty} \sum_{\substack{m,n\in\mathbb{A} \\ m\neq 0 \\ (m,n)\equiv(0,a) \mod P \\ \deg n < \deg m}} |m|^{-k} e\{-\beta x\} e\left\{\frac{\beta n}{m}\right\} |(y,x)|^{-k} dx$$

$$+ \int_{\mathcal{P}_\infty} \sum_{\substack{m,n\in\mathbb{A} \\ m\neq 0 \\ (m,n)\equiv(0,a) \mod P \\ \deg n \geq \deg m}} |m|^{-k} e\{-\beta x\} e\left\{\frac{\beta n}{m}\right\} \left|\left(y, \frac{n}{m}\right)\right|^{-k} dx.$$

The second integral is easily be seen to be zero: with $d = \deg\beta$,

$$\int_{\mathcal{P}_\infty} e\{-\beta x\} \sum_{\substack{m,n\in\mathbb{A} \\ m\neq 0 \\ (m,n)\equiv(0,a) \mod P \\ \deg n \geq \deg m}} |m|^{-k} e\left\{\frac{\beta n}{m}\right\} \left|\left(y, \frac{n}{m}\right)\right|^{-k} dx$$

$$= \sum_{x_0\in\mathcal{P}_\infty/\mathcal{P}_\infty^d} \int_{\mathcal{P}_\infty^d} e\{-\beta(x_0+x)\} \sum_{\substack{m,n\in\mathbb{A} \\ m\neq 0 \\ (m,n)\equiv(0,a) \mod P \\ \deg n \geq \deg m}} |m|^{-k} e\left\{\frac{\beta n}{m}\right\} \left|\left(y, \frac{n}{m}\right)\right|^{-k} dx \quad =$$

$$= \sum_{x_0\in\mathcal{P}_\infty/\mathcal{P}_\infty^d} e\{-\beta x_0\} \int_{\mathcal{P}_\infty^d} e\{-\beta x\} \sum_{\substack{m,n\in\mathbb{A} \\ m\neq 0 \\ (m,n)\equiv(0,a) \mod P \\ \deg n \geq \deg m}} |m|^{-k} e\left\{\frac{\beta n}{m}\right\} \left|\left(y, \frac{n}{m}\right)\right|^{-k} dx.$$

Now,

$$\sum_{x\in\mathcal{P}_\infty/\mathcal{P}_\infty^d} e\{-\beta x\} = \sum_{x_0\in\mathcal{P}_\infty/\mathcal{P}_\infty^{d-1}} e\{-\beta x_0\} \sum_{x\in\mathcal{P}_\infty^{d-1}/\mathcal{P}_\infty^d} e\{-\beta x\}.$$

The second sum is easily seen to be a complete character sum hence equal to zero since

$$\begin{aligned}\sum_{x\in\mathcal{P}_\infty^{d-1}/\mathcal{P}_\infty^d} e\{-\beta x\} &= \sum_{u\in\mathbb{F}_q} e\{-\beta u T^{1-d}\}\\ &= \sum_{u\in\mathbb{F}_q} \exp\left(\frac{-2\pi i\,\mathrm{Tr}(\beta_d u)}{p}\right)\\ &= \sum_{u'\in\mathbb{F}_q} \exp\left(\frac{-2\pi i\,\mathrm{Tr}(u')}{p}\right)\\ &= 0.\end{aligned}$$

For fixed m, the sum on n in the first integral term is a finite sum allowing that term to be written as

$$\sum_{\substack{m,n\in\mathbb{A}\\ m\neq 0\\ (m,n)\equiv(0,a) \mod P\\ \deg n<\deg m}} |m|^{-k} e\left\{\frac{\beta n}{m}\right\} \int_{\mathcal{P}_\infty} e\{-\beta x\}|(y,x)|^{-k}dx$$

$$= \sum_{\substack{P|m\\ m\neq 0\\ n \mod m}} |m|^{-k} e\left\{\frac{\beta n}{m}\right\} \int_{\mathcal{P}_\infty} e\{-\beta x\}|(y,x)|^{-k}dx.$$

In this situation $(m,P)=P$. Letting $n=sP+a$ and $m=m'P$, observe that as n runs modulo m, s runs modulo $m'=\frac{m}{P}$.

Consequently, the β^{th} coefficient is

$$\sum_{\substack{m'\in\mathbb{A}\\ m'\neq 0}} |Pm'|^{-k} \sum_{s \mod m'} e\left\{\frac{\beta(sP+a)}{m'P}\right\} \int_{\mathcal{P}_\infty} \frac{e\{-\beta x\}}{|(y,x)|^k}dx$$

$$= \sum_{\substack{m' \in \mathbb{A} \\ m' \neq 0}} |Pm'|^{-k} e\left\{\frac{\beta a}{m'P}\right\} \sum_{s \mod m'} e\left\{\frac{\beta s}{m'}\right\} \int_{\mathcal{P}_\infty} \frac{e\{-\beta x\}}{|(y,x)|^k} dx.$$

The sum on s is a character sum. So

$$\sum_{s \mod m'} e\left\{\frac{\beta a}{m'}\right\} = \begin{cases} |m'| & \text{if } m'|\beta \ ; \\ 0 & \text{otherwise} \end{cases}$$

introducing a divisibility relation $m'|\beta$. So the β^{th} coefficient is

$$|P|^{-k} \sum_{\substack{m' \in \mathbb{A} \\ m'|\beta}} |m'|^{1-k} e\left\{\frac{\beta a}{m'P}\right\} \int_{\mathcal{P}_\infty} \frac{e\{-\beta x\}}{|(y,x)|^k} dx.$$

Applying Lemma 3.2.5 the β^{th} coefficient is

$$(q-1)Q(\beta,y)|P|^{-k} \sum_{\substack{m' \in \mathbb{A}, \text{monic} \\ m'|\beta}} |m'|^{1-k} e\left\{\frac{\beta a}{m'P}\right\}$$

□

The main result for irreducible level P follows, easily expressed in the next theorem.

Theorem 3.2.7 The β^{th} coefficient of Eisenstein series of (integer)weight k, irreducible level P with character $\chi = \chi_1\chi_2$, modulo P ,

$$E_P(\tau) = \sum_{\substack{a \mod P \\ c \mod P}} \chi(c) e\left\{-\frac{ac}{P}\right\} G(\tau; P, a; P)$$

is

$$c_\beta(\tau) = Q(\beta,y)|P|^{1-k} \sum_{\substack{m\,|\beta \\ m \in \mathbb{A}, \text{ monic}}} \chi\left(\frac{\beta}{m}\right) |m|^{1-k}.$$

Proof: The β^{th} coefficient of

$$E_P(\tau) = \sum_{\substack{a \mod P \\ c \mod P}} \chi(c) e\left\{-\frac{ac}{P}\right\} G(\tau; P, a; P)$$

is

$$c_{\beta,k}(\tau) = (q-1)Q(\beta,y)|P|^{-k} \sum_{\substack{a \mod P \\ c \mod P}} \chi(c) e\left\{-\frac{ac}{P}\right\} \sum_{\substack{m|\beta \\ m \in \mathbb{A}\setminus\{0\}}} |m|^{1-k} e\left\{\frac{\beta a}{mP}\right\}$$

with $\nu = \frac{\beta}{m}$

$$c_{\beta,k}(\tau) = (q-1)Q(\beta,y)|P|^{-k} \sum_{\substack{c \mod (P), \\ \nu|\beta, \nu \neq 0}} \chi(c) \left|\frac{\beta}{\nu}\right|^{1-k} \sum_{a \mod P} e\left\{\frac{a(\nu - c)}{P}\right\}.$$

$$= (q-1)Q(\beta,y)|P|^{1-k} \sum_{\substack{c \mod P \\ c \equiv \nu \mod P \\ \nu|\beta, \nu \neq 0}} \chi(c) \left|\frac{\beta}{\nu}\right|^{1-k}$$

$$= Q(\beta,y)|P|^{1-k} \sum_{\substack{m\,|\beta \\ m \in \mathbb{A},\ \text{monic}}} \chi\left(\frac{\beta}{m}\right) |m|^{1-k}$$

□

3.3 Square-free Level N

Recalling 3.2.3, begin by defining the **G-Eisenstein series of level N**.

Definition 3.3.1 The **G-Eisenstein series of (integer)weight k, square-free level N** is defined to be

$$G(\tau; c, d; N) = \sum_{\substack{m,n \in \mathbb{A}, m \neq 0 \\ (m,n) \equiv (c,d) \mod P}} |m\tau + n|^{-k}$$

where $\tau = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$.

Definition 3.3.2 The **Eisenstein series of (integer)weight k, square-free level N with character χ modulo** N is for $D|N$

$$E_D(\tau) = \sum_{\substack{a \mod N \\ b \mod N/D \\ c \mod D}} \chi_1(b)\chi_2(c)e\{\frac{-ac}{D}\}G(\tau; bD, a; N)$$

where $\chi = \chi_1\chi_2$.

Proposition 3.3.3 The β^{th} coefficient of the Fourier expansion of G-Eisenstein series of (integer)weight k, square-free level N,

$$G(\tau; bD, a; N) = \sum_{\substack{m,n \in \mathbb{A}, m \neq 0 \\ (m,n) \equiv (bD,a) \mod N}} |m\tau + n|^{-k}$$

$$c_\beta(y) = Q(\beta, y)|D|^{-k} \sum_{\substack{m' \text{monic} \\ m'|\beta \\ m' \equiv b \mod \frac{N}{D}}} |m'|^{1-k} e\left\{\frac{\beta a}{m'D}\right\}$$

where $\tau = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$, $\quad y = T^{-2\ell}$ and $Q(\beta, y) = \left(\frac{1-q^{(d-2\ell)k}}{1-q^{-k}}\right)|y|^{2-k}|\beta|$

Proof: The proof follows similarly to that of Proposition 3.2.6. The β^{th} coefficient of $G(\tau; bD, a; N)$ is given by

$$\int_{\mathcal{P}_\infty} \sum_{\substack{m,n \in \mathbb{A} \\ m \neq 0 \\ (m,n) \equiv (bD,a) \mod N}} e\{-\beta x\}|(my, mx + n)|^{-k} dx =$$

$$\int_{\mathcal{P}_\infty} \sum_{\substack{m,n \in \mathbb{A} \\ m \neq 0 \\ (m,n) \equiv (bD,a) \mod N \\ \deg n < \deg m}} |m|^{-k} e\{-\beta x\} e\left\{\frac{\beta n}{m}\right\} |(y,x)|^{-k} dx$$

$$+ \int_{\mathcal{P}_\infty} \sum_{\substack{m,n \in \mathbb{A} \\ m \neq 0 \\ (m,n) \equiv (bD,a) \mod N \\ \deg n \geq \deg m}} |m|^{-k} e\{-\beta x\} e\left\{\frac{\beta n}{m}\right\} \left|\left(y, \frac{n}{m}\right)\right|^{-k} dx.$$

As in Proposition 3.2.6 the second term is zero.

Applying Lemma 3.2.5 the integral can be evaluated as $(q-1)Q(\beta, y)$. For fixed m the first term is a finite sum allowing that term to be written as

$$(q-1)Q(\beta, y) \sum_{\substack{m,n \in \mathbb{A}, m \neq 0 \\ (m,n) \equiv (bD,a) \mod N \\ \deg n < \deg m}} |m|^{-k} e\left\{\frac{\beta n}{m}\right\}.$$

In this situation since N is square-free $(m, D) = D$. Letting $n = s'N + a$, $t = \frac{N}{D}$, and $m = m'D$, observe that as n runs modulo m, $s = \frac{s'}{t}$ runs modulo $m' = \frac{m}{D}$. Consequently, the β^{th} coefficient is

$$(q-1)Q(\beta, y) \sum_{\substack{m' \in \mathbb{A} \\ m' \neq 0}} |Dm'|^{-k} \sum_{s \mod m'} e\left\{\frac{\beta(sD+a)}{m'D}\right\}$$

similarly introducing a divisibility relation $m'|\beta$ as in 3.2.6 and a congruence condition $m' \equiv b \mod \frac{N}{D}$. So the β^{th} coefficient is

$$Q(\beta, y)|D|^{-k} \sum_{\substack{m' \in \mathbb{A}, \text{monic} \\ m'|\beta}} |m'|^{1-k} e\left\{\frac{\beta a}{m'D}\right\}.$$

$\square$

Theorem 3.3.4 The β^{th} coefficient of Eisenstein series of (integer)weight k, square-free level N with character χ, and $D|N$

$$E_D(\tau) = \sum_{\substack{a \mod N \\ b \mod N/D \\ c \mod D}} \chi_1(b)\chi_2(c)e\left\{\frac{-ac}{D}\right\} G(\tau; bD, a; N)$$

is

$$c(\beta,\tau) = Q(\beta,y)|D|^{1-k}|N| \sum_{m|\beta,\text{monic}} \chi_1(m)\chi_2\left(\frac{\beta}{m}\right)|m|^{1-k}$$

where $Q(\beta,y) = \left(\frac{1-q^{(d-2\ell)k}}{1-q^{-k}}\right)|y|^{2-k}|\beta|$ and $\tau = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$.

Proof: The β^{th} coefficient of

$$E_D(\tau) = \sum_{\substack{a \mod N \\ b \mod N/D \\ c \mod D}} \chi_1(b)\chi_2(c)e\left\{\frac{-ac}{D}\right\} G(\tau; bD, a; N)$$

is

$$c(\beta,\tau) = (q-1)Q(\beta,y)|D|^{-k} \sum_{\substack{a \mod N \\ b \mod N/D \\ c \mod D}} \chi_1(b)\chi_2(c) \sum_{\substack{m|\beta \\ m\in\mathbb{A}\setminus\{0\}}} |m|^{1-k} e\left\{\frac{a(\frac{\beta}{m}-c)}{D}\right\}$$

Looking at the sum on a , for fixed c and m ,

$$\sum_{a \mod N} e\left\{\frac{a(\frac{\beta}{m}-c)}{D}\right\} = \begin{cases} |N|, & \text{if } c \equiv \frac{\beta}{m} \mod D\ ; \\ 0, & \text{otherwise.} \end{cases}$$

So

$$c(\beta,\tau) = (q-1)Q(\beta,y)|D|^{1-k}|N| \sum_{\substack{c \mod D \\ b \mod N/D}} \chi_1(b)\chi_2(c) \sum_{\substack{m|\beta \\ m\in\mathbb{A}\setminus\{0\}}} |m|^{1-k}$$

with the added congruence conditions that $c \equiv \frac{\beta}{m} \mod D$ and $b \equiv m \mod \frac{N}{D}$. Since χ_1 and χ_2 are characters modulo $\frac{N}{D}$ and D respectively , $\chi_1(c) = \chi_1(\frac{\beta}{m})$ and $\chi_2(b) = \chi_2(m)$. Thus

$$c(\beta, \tau) = Q(\beta, y)|D|^{1-k}|N| \sum_{m|\beta,\text{monic}} \chi_1(m)\chi_2\left(\frac{\beta}{m}\right)|m|^{1-k}$$

□

3.4 General Level N

Recalling the setting in the previous two sections, we begin by defining the **G-Eisenstein series of level N**.

Definition 3.4.1 The **G-Eisenstein series of (integer)weight k, level N** is defined to be

$$G(\tau; c, d; N) = \sum_{\substack{m,n \in \mathbb{A}, m \neq 0 \\ (m,n) \equiv (c,d) \mod P}} |m\tau + n|^{-k}$$

where $\tau = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$.

Next we define the general Eisenstein series of weight k, level N, and character χ.

Definition 3.4.2 The **Eisenstein series of (integer)weight k, level N with character** χ is

$$E_D(\tau) = \sum_{\substack{a \mod N \\ b \mod N/D \\ c \mod D}} \chi_1(b)\chi_2(c)e\left\{\frac{-ac}{D}\right\} G(\tau; bD, a; N)$$

where $\chi = \chi_1\chi_2$ and $D|N$.

Proposition 3.4.3 The β^{th} coefficient of the Fourier expansion of G-Eisenstein series of (integer)weight k, level N, and $D|N$

$$G(\tau; bD, a; N) = \sum_{\substack{m,n\in\mathbb{A}, m\neq 0 \\ (m,n)\equiv(bD,a) \mod P}} |m\tau + n|^{-k}$$

is

$$c_\beta = (q-1)Q(\beta, y)|M|^{-k} \sum_{\substack{m'|\beta \\ m'\equiv b \mod \frac{N}{M}}} |m|^{1-k} e\left\{\frac{\beta a}{mM}\right\}$$

where $\tau = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$, $\quad y = T^{-2\ell}$ and $Q(\beta, y) = \left(\frac{1-q^{(d-2\ell)k}}{1-q^{-k}}\right)|y|^{2-k}|\beta|$ and $M = (m, N)$

Proof: The β^{th} coefficient of $G(\tau; bD, a; N)$ is given by

$$\int_{\mathcal{P}_\infty} \sum_{\substack{m,n\in\mathbb{A} \\ m\neq 0 \\ (m,n)\equiv(bD,a) \mod N}} e\{-\beta x\}|(my, mx+n)|^{-k}dx =$$

$$\int_{\mathcal{P}_\infty} \sum_{\substack{m,n\in\mathbb{A} \\ m\neq 0 \\ (m,n)\equiv(bD,a) \mod N \\ \deg n<\deg m}} |m|^{-k} e\{-\beta x\} e\left\{\frac{\beta n}{m}\right\} |(y,x)|^{-k}dx$$

$$+ \int_{\mathcal{P}_\infty} \sum_{\substack{m,n\in\mathbb{A} \\ m\neq 0 \\ (m,n)\equiv(bD,a) \mod N \\ \deg n\geq\deg m}} |m|^{-k} e\{-\beta x\} e\left\{\frac{\beta n}{m}\right\} \left|\left(y, \frac{n}{m}\right)\right|^{-k} dx.$$

The second term is zero as in Proposition 3.2.6 in the prior section 3.2. Applying Lemma 3.2.5 the integral can be evaluated as $(q-1)Q(\beta, y)$. For fixed m the first term is a finite sum allowing that term to be written as

$$(q-1)Q(\beta,y) \sum_{\substack{m,n\in\mathbb{A},m\neq 0\\(m,n)\equiv(bD,a) \mod N\\ \deg n<\deg m}} |m|^{-k} e\left\{\frac{\beta n}{m}\right\}.$$

In this situation let $(m,N)=M$, then $m=Mm'$ and for $n=N\ell+a=M(N'\ell)+a$ and now $\left(N',\frac{m}{M}\right)=1$, observe that as n runs modulo m, $s=\frac{s'}{t}$ runs modulo $m'=\frac{m}{M}$. The congruence condition on m, $m\equiv bD$ now yields the congruence $m'\equiv b \mod \frac{N}{D}$ for m'.Consequently, the β^{th} coefficient is

$$(q-1)Q(y)\sum_{\substack{m'\in\mathbb{A}\\ m'\neq 0}} |Mm'|^{-k} \sum_{s \mod m'} e\left\{\frac{\beta(sM+a)}{m'M}\right\}$$

similarly introducing a divisibility relation $m'|\beta$ again as in Proposition 3.2.6. So the β^{th} coefficient is

$$(q-1)Q(y)|M|^{-k} \sum_{\substack{m'\in\mathbb{A},\\ m'|\beta\\ m'\equiv b \mod \frac{N}{D}}} |m'|^{1-k} e\left\{\frac{\beta a}{m'M}\right\}.$$

□

Theorem 3.4.4 Let

$$E_D(\tau)=\sum_{\substack{a \mod N\\ b \mod N/D\\ c \mod D}} \chi_1(b)\chi_2(c)e\left\{\frac{-ac}{D}\right\}G(\tau;bD,a;N)$$

be the Eisenstein series of (integer)weight k, level N with character χ, modulo N, with

$\chi = \chi_1\chi_2$ and $D|N$. Then its β^{th} Fourier coefficient is

$$c(\beta,\tau) = Q(\beta,y)|M|^{-k}|N|^2 \sum_{\substack{m' \in \mathbb{A},\, \text{monic} \\ m'|\beta}} \chi_1(sm')\chi_2\left(\frac{\beta}{sm'}\right)|m'|^{1-k},$$

where $Q(\beta,y) = \left(\frac{1-q^{(d-2\ell)k}}{1-q^{-k}}\right)|y|^{2-k}|\beta|, \tau = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}, \quad y = T^{-2\ell}, M = (m,N), \deg\beta = d$, and $s = \frac{M}{D}$.

Proof: As in the previous Proposition 3.4.3 let $M = (m,N)$ and $m = m'M$. The β^{th} coefficient of

$$E_D(\tau) = \sum_{\substack{a \mod N \\ b \mod N/D \\ c \mod D}} \chi_1(b)\chi_2(c)e\{\frac{-ac}{D}\}G(\tau; bD, a; N)$$

is

$$c(\beta,\tau) = (q-1)Q(\beta,y)|M|^{-k} \sum_{\substack{a \mod N \\ b \mod N/D \\ c \mod D}} \chi_1(b)\chi_2(c)e\left\{\frac{-ac}{D}\right\} \sum_{\substack{m|\beta \\ m' \in \mathbb{A}\setminus\{0\}}} |m'|^{1-k}e\left\{\frac{\beta a}{m'M}\right\}$$

$$= (q-1)Q(\beta,y)|M|^{-k} \sum_{\substack{a \mod N \\ b \mod N/D \\ c \mod D}} \chi_1(b)\chi_2(c) \sum_{\substack{m'|\beta \\ m' \in \mathbb{A}\setminus\{0\}}} |m'|^{1-k}e\left\{\frac{a(\frac{\beta}{sm'}-c)}{D}\right\}.$$

As before by looking at the sum on a, note that

$$\sum_{a \mod N} e\left\{\frac{a(\frac{\beta}{sm'}-c)}{D}\right\} = \begin{cases} |N|, & \text{if } c \equiv \frac{\beta}{sm'} \mod D\,; \\ 0, & \text{otherwise.} \end{cases}$$

Thus the coefficient $c(\beta,\tau)$ is

$$c(\beta,\tau) = (q-1)Q(\beta,y)|M|^{-k}|N| \sum_{\substack{b \mod N/D \\ c \mod D \\ c \equiv \frac{\beta}{sm'} \mod D}} \chi_1(b)\chi_2(c) \sum_{\substack{m'|\beta \\ m' \neq 0}} |m'|^{1-k}$$

Applying the congruence conditions on b and c to $\chi_1(b)$ and $\chi_2(c)$ the resulting coeeficient is

$$c(\beta,\tau) = Q(\beta,y)|M|^{-k}|N|^2 \sum_{\substack{m' \in \mathbb{A},\,\text{monic} \\ m'|\beta}} \chi_1(sm')\chi_2\left(\frac{\beta}{sm'}\right)|m'|^{1-k}$$

□

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