Simon’s fundamental rich-gets-richer model entails a dominant first-mover advantage

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Herbert Simon’s classic rich-gets-richer model is one of the simplest empirically supported mechanisms capable of generating heavy-tail size distributions for complex systems. Simon argued analytically that a population of flavored elements growing by either adding a novel element or randomly replicating an existing one would afford a distribution of group sizes with a power-law tail. Here, we show that, in fact, Simon’s model does not produce a simple power law size distribution as the initial element has a dominant first-mover advantage, and will be overrepresented by a factor proportional to the inverse of the innovation probability. The first group’s size discrepancy cannot be explained away as a transient of the model, and may therefore be many orders of magnitude greater than expected. We demonstrate how Simon’s analysis was correct but incomplete, and expand our alternate analysis to quantify the variability of long term rankings for all groups. We find that the expected time for a first replication is infinite, and show how an incipient group must break the mechanism to improve their odds of success. Our findings call for a reexamination of preceding work invoking Simon’s model and provide a revised understanding going forward.

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Across the spectrum of natural and constructed phenomena, descriptions of the architecture and dynamical behavior of complex systems repeatedly involve heavy-tailed distributions. For systems involving components of variable size $S$, many bear size distributions with power-law decays of the form $Pr(S) \sim S^{-\gamma}$ [1, 2]: word usage frequency in language [3], the number of species per genus [1, 4], citation numbers for scientific papers [5], node degree in networks [6, 8], firm sizes [9], and the extent of system failures such as forest fires [10, 11].

Elucidating and understanding the most essential dynamical models leading to power-law size distributions is an essential task. While the mechanisms giving rise to such distributions are diverse, they generally involve growth and replication.

In his famous 1955 paper on skewed distributions [1], Simon built on classical urn model theory to show that an elementary, single parameter, rich-gets-richer mechanism could lead a growing population to produce a pure power-law size distribution of groups of like-flavored elements. Simon’s model, detailed below, is governed by an innovation probability $\rho$ which Simon argued controls the size distribution exponent as $\gamma = 1 + 1/(1 - \rho)$, and, equivalently, the Zipf’s law exponent $\alpha = 1 - \rho$.

In this letter, we show analytically and through simulations that Simon’s analysis was strikingly incomplete: the initial group enjoys a profound “first-mover advantage” on the order of the inverse of the innovation probability, $1/\rho$. This is not a small correction to a long established theory. As the innovation probability is typically close to 0, the initial group’s size may be orders of magnitude greater than would be consistent with a simple power law. Nor, as we will show, can the first group be dismissed as a transient or as a kind of null group and not part of the system.

In what follows, we first describe Simon’s model, present results from simulations, analytically determine the first-mover’s advantage, explore the detailed long term dynamics of all groups, and, finally, consider the implications for understanding real-world, rich-gets-richer systems.

The algorithm for Simon’s classic rich-gets-richer model is simple. We are concerned with the growth of a population of elements where each element has a flavor, and elements of the same flavor form a group. In modeling real systems, flavors may represent a city, a word, or the destination of a link in a network [1, 12]. Beginning with a single element at time $t = 1$, an element is added to the population at each discrete time step $t \geq 2$. Representing innovation, the arriving element has, with probability $\rho$, a new, previously unseen flavor. Alternately, with probability $1 - \rho$, the arriving element is a replication taking on the flavor of a randomly chosen existing element. For a more general version, we will also later allow for a dynamic innovation probability $\rho_t$, as Simon also did in [1].

In Fig. 1, we provide visualizations along with Zipf-like distributions and size distributions for Simon’s model for $\rho = 0.1, 0.01, 0.001$, and 0.0001. In Figs. A–D, we show results for four sample simulations corresponding to these values of $\rho$ after $10^6$ time steps. Each disc rep-
Fig. 1. Results from simulations of Simon’s model showing an inherent dominant first-mover advantage. A–D. Visualizations of group sizes after approximately \( t = 10^6 \) time steps for \( \rho = 0.1, 0.01, 0.001, \) and 0.0001. Group sizes are proportional to disc area. The colors are a function of each simulation’s specific history but do match across groups of the same size within each system. E–H. Main plots: Zipf-like distributions for an ensemble of 200 simulations in gray and theory in red [see Eq. [5]]. Differing from the simulations for the visualizations, these were run until \( N(0) = 10^5 \), approximately \( 10^5 / \rho \) time steps. The dark gray curve indicates the median size of the \( n \)th arriving group and the light gray bounds 2.5 to 97.5 percentiles. For clarity, the median sizes of the first and second groups are highlighted as \( \log_{10} S_1 \) and \( \log_{10} S_2 \). Insets: For each value of \( \rho \), we show the size distribution for a single simulation, again indicating \( \log_{10} S_1 \). Note that the exponents for the Zipf distribution and size distributions are connected as \( \gamma = 1 + 1/\alpha \). We provide interactive simulations for Simon’s model at [http://compsstorylab.org/share/papers/dodds2016b/](http://compsstorylab.org/share/papers/dodds2016b/).

represents a group with area proportional to group size. To the eye, for \( \rho = 0.1 \), the first group appears to be somewhat outsized but perhaps not inconsistent. However, for next three (decreasing) values of \( \rho \), the first group is evidently different, increasingly accruing the bulk of all new elements.

In Figs. E–F, we make clear the first-mover advantage through Zipf-like distributions for group size (main plots) and size distributions (inset plots) for the same ordering of \( \rho \) values. The group sizes in the main plots are a function of group arrival number rather than rank according to decreasing size (hence Zipf-like), and come from an ensemble of simulations (median in dark gray, 2.5 to 97.5 percentile range in light gray; see caption for details) and overlaid theory (red). We examine the variation in group size with group arrival order later on. As per Simon’s analysis, median group size \( S_n \) behaves as \( n^{-\alpha} \) for \( n \geq 10 \) where \( \alpha (= 1 - \rho) \) is the Zipf exponent. However, as we demonstrate theoretically below, the dominant group \( S_1 \) is larger than would be expected by a factor of \( 1/\rho \).

Again for single example simulations, the inset size distributions in Figs. E–F show the same disparity with the largest element isolated from the main power-law-obeying size distribution. We can now see why Simon’s analysis, while technically correct, fell short: The size distribution he derived fit all but one point which, if not observed and handled appropriately, vanishes in contribution in the infinite system size limit.

To understand this first-mover advantage, we carry out a re-analysis of Simon’s model. New groups are initiated stochastically with the \( n \)th group first appearing on average at time \( t_{\text{init}}^{(n)} \). We write the number of elements in the \( n \)th group at time \( t \geq t_{\text{init}}^{(n)} \) as \( S_{n,t} \), and each group starts with a single element: \( S_{n,t_{\text{init}}^{(n)}} = 1 \) (we examine this choice later). We assume an initial condition of a population of 1 element at time \( t = 1 \). If at time \( t \), we have \( N_t \) distinct groups, the probability that a randomly drawn element belongs to the \( n \)th group is then:

\[
P_{n,t} = \frac{S_{n,t}}{\sum_{n'=1}^{N_t} S_{n',t}} = \frac{S_{n,t}}{t}.
\]  

We construct an evolution equation for the size of the
nth arriving group, \( S_{n,t} \). At time \( t \), for the \( n \)th group to increase in number by 1, replication must be chosen (occurring with probability \( 1 - \rho_t \)), and then an element in the \( n \)th group must be replicated, leading to the probabilistic statement:

\[
(S_{n,t+1} - S_{n,t}) = (1 - \rho_t) \cdot \frac{S_{n,t}}{t} \cdot (+1).
\]  

In the case of Simon’s fundamental model with a fixed innovation rate, \( \rho_t = \rho \), we proceed with a difference equation calculation. Our primary analysis is approximate as we drop the expectation on the left hand side of Eq. (2).

For fixed \( \rho_t = \rho \) and shifting from \( t \) to \( t - 1 \), Eq. (2) gives the approximation, for \( t \geq t_{\text{init}} \),

\[
S_{n,t} = \left[ 1 + \frac{(1 - \rho)}{t - 1} \right] S_{n,t-1}. 
\]  

Again given that \( S_{n,t_{\text{init}}} = 1 \), we have

\[
S_{n,t} = \left[ 1 + \frac{(1 - \rho)}{t - 1} \right] \left[ 1 + \frac{(1 - \rho)}{t - 2} \right] \cdots \left[ 1 + \frac{(1 - \rho)}{t_{\text{init}} - 1} \right] \cdot 1
\]

\[
= \frac{\Gamma(t + 1 - \rho) \Gamma(t_{\text{init}})}{\Gamma(t_{\text{init}} + 1 - \rho) \Gamma(t)} = \frac{B(t_{\text{init}}, 1 - \rho)}{B(t, 1 - \rho)},
\]  

where \( B \) is the beta function. Our analysis thus far has been heading towards the same conclusion as Simon’s which took a different route following the evolution of the system’s size distribution. Now, however, we find a distinction that renders the initial group special. For constant \( \rho \ll 1 \), the \( n \)th group first arrives, on average, at \( t_{\text{init}} \sim \frac{n^{-1}}{\rho} \) where \([ \cdot ]\) is the rounding operator. But this is only valid for \( n \geq 2 \) because \( t_{\text{init}} = 1 \). For large \( t \), we find the size of the \( n \)th arriving group to be:

\[
S_{n,t} \sim \begin{cases} 
\frac{1}{1 - (1 - \rho)} \left[ \frac{1}{t} \right]^{-\alpha} & \text{for } n = 1, \\
\rho^{1-\rho} \left[ \frac{2n^{-1}}{1-t} \right]^{-\alpha} & \text{for } n \geq 2.
\end{cases}
\]  

with real-world data [1][2], the first group is an outlier clearly separated from the power-law size distribution.

We note that the first group may be interpreted as a so-called “dragon-king” [13] but is here entirely a product of an elementary rich-get-richer model and not an exogenous mechanism singularly affecting the largest events within a system.

We now delve into how arrival order of groups \( (n) \) relates to final rank \( (r) \), an issue arising with the Zipf-like distributions we presented in Fig. [1] and elided in our preceding analysis. While it is evident that Simon’s model must produce a degree of indeterminacy in ranking, our goal here is largely to use simulations to characterize the nature and extent of rank variability, leaving possible theoretical work for the future. In the main plot of Fig. [2A], we show results for a single simulation with \( \rho = 0.01 \), and \( 10^7 \) total elements (equivalent to run time), and \( N(t) \sim 10^9 \) distinct groups. The dark points indicate the actual size of the \( n \)th group while the blue curve is the resulting, properly ranked Zipf distribution. While the first mover dominates as expected, and the second arriving group obtains the second ranking in this instance, we observe substantial and increasing decoupling between arrival order and final rank as the system grows.

To further explore this decoherence, we generate an ensemble of 1000 simulations of the same system. The effect on final group ordering brought about by decreasing \( \rho \) saturates quickly and all of the following results regarding arrival time and rank are essentially the same for \( \rho \leq 0.01 \) with little quantitative change. We first measure the variability of the resulting distributions using the Jenson-Shannon divergence (JSD), time-ordered against time-ordered and rank-ordered against rank-ordered, a total of \( ^4C_2 \) such comparisons for each. In the inset to Fig. [2A], we show the distributions of JSD for each ordering, finding a typical disparity for the time-ordered size distributions of 0.25 bits.

In the main plot of Fig. [2B], we present overall group rank \( r \) as a function of group arrival number \( n \) for our ensemble of 1000 simulations. The pale blue line indicates the median, and the surrounding gray regions mark the 25 to 75 percentile range, the 2.5 to 97.5 percentile range, and the minimum to maximum range. In the inset, we rescale and collapse the final rank distribution for arrival group numbers 10\(^2\) to 10\(^3\) based on Eq. (5). We see that from around just the 10th arriving group on, 95% of a group’s final rank spans a remarkable two orders of magnitude around the median \( r = n \), skewed towards higher values as shown in the inset. Thus, while by equating rank and arrival number, our analysis is effective for the median size of groups, the system’s specific dynamics are considerably more complex.

Lastly, in Fig. [2C], we examine the distribution of possible group sizes as a function of group arrival number \( n \). Rather than possessing a single maximum that increases with \( n \), we find an exponential distribution is a good approximation. The inset gives three example distributions and the main plot shows an appropriate rescaling
of all 100 ≤ n ≤ 1000.

For all but a small initial collection of groups, the mode group size is thus close to 1, consistent with Simon’s asymptotic result that 1/(2 − ρ) ≈ 1/2 of all groups contain only one element (hapax legomena for texts). To understand how half of all initiated groups never grow in size, we determine the probability of a group gaining a new member as a function of its arrival time and current size. The probability that the nth arriving group, which is of, say, size $S_{n,t} = k$ at time $t$, fails to replicate for all of times $t$ through $t + τ − 1$ before replicating at time $t + τ$ is:

$$
\text{Pr}( S_{n,t+i} = k+1 | S_{n,t+i} = k \text{ for } i = 0, \ldots, τ - 1) = \prod_{i=0}^{τ-1} \left[ 1 - \frac{1+(1-ρ)\frac{k}{t+i}}{t+i} \right] \cdot \left(1-\frac{k}{t+τ} \right) \frac{k}{t+τ} = k \frac{B(τ, t)}{B(τ, t -(1-ρ))} \frac{1-\rho}{t+τ} \approx \frac{τ^{-(1-ρ)k}}{t+τ},
$$

where we have again used that $B(x, y) \sim \Gamma(y)x^{-y}$ for large $x$. We observe a power-law decay with two scaling regimes. For $τ \gg t$, the probability behaves as $τ^{-(1-ρ)k}$. As the nth arriving group starts with $S_{n,0} = 1$ element, the exponent is $-(2 − ρ) > -2$ and the expected time for replication is infinite. Once a group has replicated, the expected time becomes finite. A newly arriving group is therefore greatly advantaged if it can step out of the mechanism and begin with even just two elements.

We close our letter with some thoughts on how the dominant first-mover advantage of Simon’s model and the variability of ordering may confound empirical analyses of real-world systems, and offer some potential resolutions.

As an idealized process, Simon’s model evidently and purposefully fails to involve many aspects and details of real world systems. Nevertheless, we must address the issue that Simon’s rich-gets-richer model has performed extremely well in analyses of real systems that do not exhibit a dominant group. Particular successes have been found in measures of the innovation probability $ρ$ and the Zipf exponent $α$ with the expectation $α = 1 - ρ \ [14]$, as well as in the fraction of groups with one or two members $[15, 17, 12]$

In moving away from a pure Simon model, we consider several variant conceptions.

One possibility is to treat the first group as being of a different kind to those that come after by declaring the first element to be a kind of unobservable null element—its selection and replication at time $t$ represents a failure of the system to produce any visible element. Given our analysis above, Zipf’s law would shift with $n → n-1$ and then be closely approximated by a power law decay $[14]$. However, this is an awkward mechanism as it requires the null element to be both hidden and different and yet visible and substantive at the same time $[15]$. This mechanism is not the same as the one formed by adding an overall master update probability to Simon’s model (i.e., at each time step, engage the element-adding mechanism according to some fixed probability $ρ_{add}$). Such a modification would only serve to slow the dynamics. Further, starting the original model with no element at time $t = 1$ would also fail for as soon as the first element appears,
the model would then act in the same way. The first-mover advantage of Simon’s model cannot be dismissed by any meaningful reinterpretation of the mechanism.

A second modification would be to allow the innovation probability to vary with time (something Simon considered in [1]; see also [16, 17]). To obtain a power-law tail for Zipf’s law, the innovation rate \( \rho \) must tend towards a constant (or nearly so). But early on, a high innovation rate would suppress the first group, and smooth out the overall distribution. A dynamic \( \rho \) is plausible for real world systems, and while in principle \( \rho \) could be estimated from data, great care would have to be taken given the stochastic evolution of a single run of a pure Simon model that we have demonstrated above.

Finally, and perhaps more realistically, we may have a system that is initially configured by an entirely different growth mechanism up until some time \( t_0 \) at which a pure Simon model takes over. Building on our earlier analysis, we can instantiate a simple version of such a system with an initial \( t = 1 \) condition of \( S_{n,1}^{\text{init}} \) elements spread over \( n_{\text{init}} \) groups ranked by size (\( S_{n,1} \geq S_{n+1,1} \) for \( 1 \leq n \leq n_{\text{init}} \)). Allowing the basic rich-gets-richer mechanism to then go into effect, the same approach of Eqs. 2–5 gives us, for large \( t \), the approximate result:

\[
S_{n,t} = \frac{B(t_{\text{init}}, 1 - \rho)}{B(S_{n,1}^{\text{init}} + t - 1, 1 - \rho)} S_{n,t_{\text{init}}}
\]

\[
\sim \begin{cases} 
\frac{S_{n,1}}{t^{(1-\rho)}} \left[ \frac{1}{S_{n,1}^{\text{init}} + t - 1} \right]^{-(1-\rho)} & \text{for } 1 \leq n \leq n_{\text{init}}, \\
\rho^{1-\rho} \left[ \frac{n_{\text{init}}}{S_{n,1}^{\text{init}} + t - 1} \right]^{-(1-\rho)} & \text{for } n > n_{\text{init}},
\end{cases}
\tag{7}
\]

which reduces to Eq. 3 if \( n_{\text{init}} = S_{n,1}^{\text{init}} = S_{1,1} = 1 \). Eq. 7 shows how the first-mover advantage is distributed across the groups present when Simon’s model is introduced, and quantifies how the likelihood for newly arriving groups to replicate diminishes as the population size of the initial elements increases. For real-world systems with Zipf’s laws that have clean power-law tails leading out from difficult-to-characterize forms for the largest groups, Eq. 7 offers a possible fit that would also estimate a time of transition from an establishing mechanism to a rich-get-richer one.

In sum, we have shown through simulations and analysis that Simon’s fundamental rich-get-richer model—now 60 years old—carries an intrinsic first-mover advantage. The first group’s size in the idealized model is outsized by a factor of \( 1/\rho \), potentially several orders of magnitude. Any attempt to attribute Simon’s mechanism to the growth of real-world systems must take into account this potentially dominant feature of the model, along with the complications of the variability of final rankings for later arriving groups, and past work must come under a new scrutiny.

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[15] One possible exception may be language evolution, with whitespace framed as a kind of null element [14].