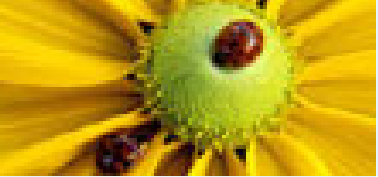


Week 8

Integer Programming: Algorithms - 2

OPR 992

Applied Mathematical Programming

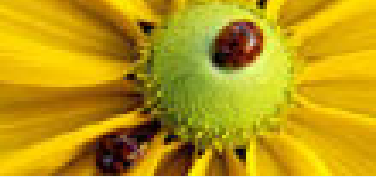


Cutting Plane Algorithms

- Introduction
- Valid Inequalities
- Chvatal-Gomory Procedure
- Gomory's Fractional Cutting Plane Algorithm

Lagrangian Duality

Cutting Plane Algorithms



Introduction

Cutting Plane Algorithms

● Introduction

- Valid Inequalities
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Lagrangian Duality

Consider the general integer program:

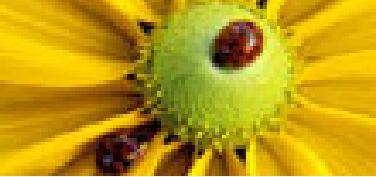
$$(IP) : \max \{c^T x : x \in X\}$$

where

$$X = \{x : Ax \leq b, x \in Z_+^n\}.$$

Proposition: $\text{conv}(X) = \{x : \tilde{A}x \leq \tilde{b}, x \geq 0\}$ is a polyhedron.

Valid Inequalities



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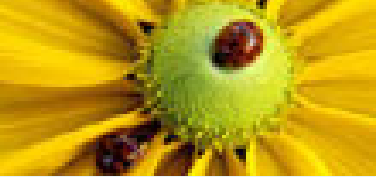
Lagrangian Duality

Definition: An inequality $\pi^T x \leq \pi_0$ is a valid inequality for $X \subseteq \mathbb{R}^n$ if $\pi^T x \leq \pi_0$ for all $x \in X$.

Examples:

- A 0-1 Knapsack Problem with weights of (3, 4, 5, 2) and a maximum weight of 7
- $X : \{(x, y) : x \leq 9999y, 0 \leq x \leq 5, y \in \{0, 1\}\}$
- $X : \{(x, y) : x \leq 10y, 0 \leq x \leq 14, y \in \mathbb{Z}_+\}$
- $X : \{x \in \mathbb{Z}_+^4 : 13x_1 + 20x_2 + 11x_3 + 6x_4 \geq 72\}$
- $X : \{x \in \mathbb{Z}_+^4, y \in \mathbb{R}_+ : 13x_1 + 20x_2 + 11x_3 + 6x_4 + y \geq 72\}$

Chvatal-Gomory Procedure



Cutting Plane Algorithms

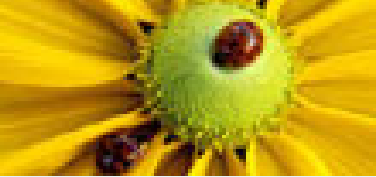
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Lagrangian Duality

Let $X = P \cap Z^n$, where $P = \{x \in \mathbb{R}_+^n : Ax \leq b\}$. $A \in \mathbb{R}^{m \times n}$ with columns $\{a_1, a_2, \dots, a_n\}$, and $u \in \mathbb{R}_+^m$. Then:

1. The inequality $\sum_{j=1}^n u^T a_j x_j \leq u^T b$ is valid for P .
2. The inequality $\sum_{j=1}^n \lfloor u^T a_j \rfloor x_j \leq u^T b$ is valid for P .
3. The inequality $\sum_{j=1}^n \lfloor u^T a_j \rfloor x_j \leq \lfloor u^T b \rfloor$ is valid for X .

Gomory's Fractional Cutting Plane Algorithm



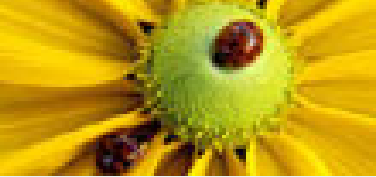
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Example:

$$\begin{array}{ll} \max & 4x_1 - x_2 \\ \text{s.t.} & 7x_1 - 2x_2 \leq 14 \\ & x_2 \leq 3 \\ & 2x_1 - 2x_2 \leq 3 \\ & x \in Z_+^2. \end{array}$$

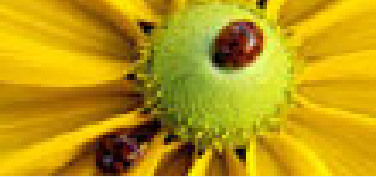


Lagrangian Duality

- Lagrangian Relaxation
- Lagrangian Dual
- Examples
- How Good is the Upper Bound?
- Solving the Lagrangian Dual
- Choosing a Lagrangian Dual

Lagrangian Duality

Lagrangian Relaxation



Consider the IP

$$\begin{aligned} z = \max \quad & c^T x \\ \text{s.t.} \quad & Dx \leq d \\ & x \in X, \end{aligned}$$

where $Dx \leq d$ are m complicating constraints.

Define the relaxation

$$\begin{aligned} z(u) = \max \quad & c^T x + u^T (d - Dx) \\ \text{s.t.} \quad & x \in X, \end{aligned}$$

where $u \geq 0$.

Lagrangian Dual

Cutting Plane Algorithms

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The Lagrangian Dual Problem:

$$w_{LD} = \min\{z(u) : u \geq 0\}$$

u are the Lagrange multipliers.

Let $u \geq 0$, if the following conditions hold:

1. $x(u)$ is an optimal solution of $IP(u)$
 2. $Dx(u) \leq d$
 3. $(Dx(u))_i = d_i$ whenever $u_i > 0$ (complementarity)
- then $x(u)$ is optimal in IP.

Examples

Cutting Plane Algorithms

Lagrangian Duality

● Lagrangian Relaxation

● Lagrangian Dual

● Examples

● How Good is the Upper Bound?

● Solving the Lagrangian Dual

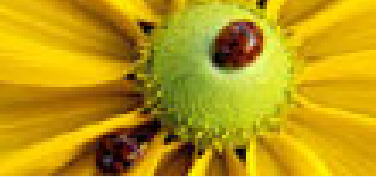
● Choosing a Lagrangian Dual

- Uncapacitated facility location: An instance with $m = 6$ clients and $n = 5$ potential locations, fixed location costs $f = (2, 4, 5, 3, 3)$ and the client-location profit matrix

$$c_{ij} = \begin{bmatrix} 6 & 2 & 1 & 3 & 5 \\ 4 & 10 & 2 & 6 & 1 \\ 3 & 2 & 4 & 1 & 3 \\ 2 & 0 & 4 & 1 & 4 \\ 1 & 8 & 6 & 2 & 5 \\ 3 & 2 & 4 & 8 & 1 \end{bmatrix}$$

- Symmetric TSP: An instance of STSP with edge cost matrix

$$c_e = \begin{bmatrix} - & 30 & 26 & 50 & 40 \\ - & - & 24 & 40 & 50 \\ - & - & - & 24 & 26 \\ - & - & - & - & 30 \\ - & - & - & - & - \end{bmatrix}$$



How Good is the Upper Bound?

Cutting Plane Algorithms

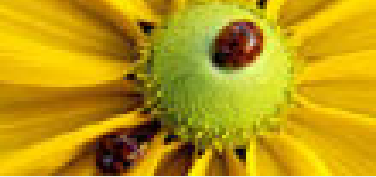
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Theorem: $w_{LD} = \max\{c^T x : Dx \leq d, x \in \text{conv}(X)\}$.

If $X = \{x \in Z_+^n : Ax \leq b\}$ and $\text{conv}(X) = \{x \in R_+^n : Ax \leq b\}$,
then $w_{LD} = \max\{c^T x : Dx \leq d, Ax \leq b, x \in R_+^n\}$.

Solving the Lagrangian Dual



Cutting Plane Algorithms

Lagrangian Duality

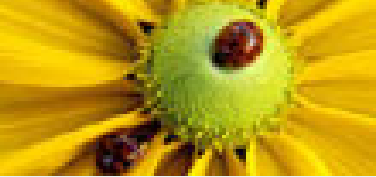
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Definition: A subgradient at u of a convex function $f : \mathbb{R}^m \rightarrow \mathbb{R}$ is a vector $\gamma(u) \in \mathbb{R}^m$ such that $f(v) \geq f(u) + \gamma(u)^T (v - u)$ for all $v \in \mathbb{R}^m$.

The Subgradient Algorithm:

1. Pick initial u^0 , let $k = 0$.
2. Solve the Lagrangian problem $IP(u^k)$ with optimal solution $x(u^k)$.
3. $u^{k+1} = \max\{u^k - \mu_k^T (d - Dx(u^k)), 0\}$.
4. Let $k = k + 1$. Go back to step 2.

Choosing a Lagrangian Dual



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Given a problem of the form

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & A_1 x \leq b_1 \\ & A_2 x \leq b_2 \\ & x \in Z_+^n, \end{array}$$

which constraint(s) should we dualize?

The choice depends on

1. The strength of the resulting Lagrangian bound
2. Ease of solution of the Lagrangian subproblems
3. Ease of solution of the Lagrangian dual problem