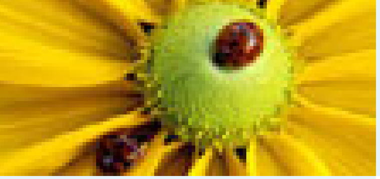
Week 7

Integer Programming: Optimality Conditions and Algorithms

OPR 992

Applied Mathematical Programming



Optimality and Bounds

$$z = \max\{c(x) : x \in X \subseteq Z^n\}$$

Find a decreasing sequence of upper bounds

$$\overline{z_1} > \overline{z_2} > \ldots > \overline{z_s} \geq z$$

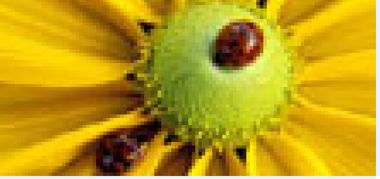
and an increasing sequence of lower bounds

$$\underline{z_1} > \underline{z_2} > \ldots > \underline{z_t} \leq z$$

such that

$$\overline{z_s} - \underline{z_t} \leq \epsilon.$$

- *Primal Bound*: Any feasible solution will provide a lower bound.
- *Dual Bound*: Any optimal solution to a relaxation will provide an upper bound.

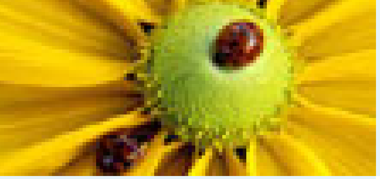


Primal Bounds: Greedy Heuristic

- Construct an initial feasible solution (incumbent) from scratch
- Focus on immediate rewards

The Knapsack Problem: Choose among 4 items, with values (12, 8, 17, 11) and weights (4, 3, 7, 5), to place in a knapsack that can accommodate a maximum weight of 9.
The Travelling Salesman Problem: Find the minimum distance tour given the distance matrix:

$$\begin{bmatrix} - & 9 & 2 & 8 & 12 & 11 \\ & - & 7 & 19 & 10 & 32 \\ & & - & 29 & 18 & 6 \\ & & & - & 24 & 3 \\ & & & & - & 19 \\ & & & & & - \end{bmatrix}$$



Primal Bounds: Local Search

- Define a neighborhood of solutions close to the incumbent
- If the best solution in the neighborhood is better than the incumbent, replace the incumbent and repeat.

Uncapacitated Facility Location: 6 clients, 4 depots, fixed costs of depots (21, 16, 11, 24) and variable costs

$$\begin{bmatrix} 6 & 2 & 3 & 4 \\ 1 & 9 & 4 & 11 \\ 15 & 2 & 6 & 3 \\ 9 & 11 & 4 & 8 \\ 7 & 23 & 2 & 9 \\ 4 & 3 & 1 & 5 \end{bmatrix}$$

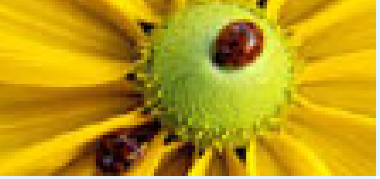
Graph Equipartition Problem: 6 nodes with edges (1,4), (1,6), (2,3), (2,5), (2,6), (3,4), (3,5), and (4,6). Find the two partitions of equal size to minimize the number of edges in between

Dual Bounds: Relaxations

Definition: A problem (RP) $z^R = \max\{f(x) : x \in T \subseteq \mathbb{R}^n\}$ is a relaxation of (IP) $z = \max\{c(x) : x \in X \subseteq \mathbb{R}^n\}$ if:

- $X \subseteq T$, and
- $f(x) \geq c(x)$ for all $x \in X$.

Proposition: If RP is a relaxation of IP, $z^R \geq z$.



Dual Bounds: LP Relaxations

Remove the integrality requirement from

$$
\begin{array}{lrcl}
\text{max} & c^T x & & \\
\text{s.t.} & Ax & \leq & b \\
 & x & \geq & 0 \\
 & x & \in & Z^n .
\end{array}
$$

Example:

$$
\begin{array}{lrcl}
\text{max} & 4x_1 - x_2 & & \\
\text{s.t.} & 7x_1 - 2x_2 & \leq & 14 \\
 & x_2 & \leq & 3 \\
 & 2x_1 - 2x_2 & \leq & 3 \\
 & x & \in & Z^n_+
\end{array}
$$

Dual Bounds: Lagrangian Relaxation

Remove the linear constraints from

$$\begin{array}{rrcl} z = \max & c^T x & & \\ \text{s.t.} & Ax & \leq & b \\ & x & \geq & 0 \\ & x & \in & Z^n \end{array}$$

and put them in the objective function

$$\begin{array}{rrcl} z(u) = \max & c^T x + u^T (b - Ax) & & \\ \text{s.t.} & x & \geq & 0 \\ & x & \in & Z^n. \end{array}$$

Proposition: $z(u) \geq z$ for all $u \geq 0$.

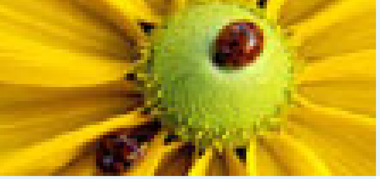
Duality for Integer Linear Programs

Definition: The two problems

$$\begin{aligned} (IP): \quad z &= \max\{c(x) : x \in X\} \\ (D): \quad w &= \min\{w(u) : u \in U\} \end{aligned}$$

form a weak-dual pair if $c(x) \leq w(u)$ for all $x \in X$ and all $u \in U$. When $z = w$, they form a strongly-dual pair.

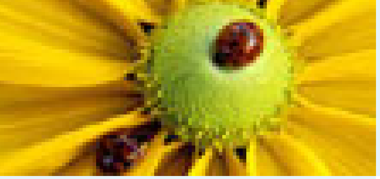
Proposition: The integer program $z = \max\{c^T x : Ax \leq b, x \in Z^n_+\}$ and the linear program $w^{LP} = \min\{b^T u : A^T u \geq c, u \in \mathbb{R}^m_+\}$ form a weak-dual pair.



Branch and Bound

An Example

$$
\begin{array}{lllll}
\text{maximize} & 17x_1 + 12x_2 & & \\
\text{subject to} & 10x_1 + 7x_2 & \leq & 40 \\
 & x_1 + x_2 & \leq & 5 \\
 & x & \in & Z_+^2
\end{array}
$$



Pruning the Branch and Bound Tree

- Pruning by Optimality
- Pruning by Bound
- Pruning by Infeasibility

Practical Issues

- Storing the tree
- Choosing a node: Depth-First vs. Best-Node First