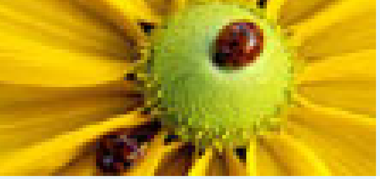
Week 3

Basics of Constrained Optimization

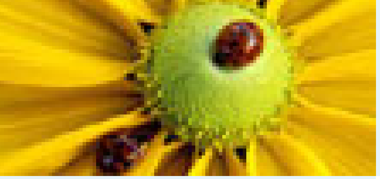
OPR 992

Applied Mathematical Programming



Problem Formulation

$$
\begin{aligned}
\text{min} \quad & f(x) & & \\
\text{s.t.} \quad & g(x) & \geq & \quad 0 \\
& h(x) & = & \quad 0.
\end{aligned}
$$



Optimality Conditions

Karush-Kuhn-Tucker Conditions

$$
\begin{aligned}
\nabla f(x) - \nabla g(x)u + \nabla h(x)v &= 0 \\
g(x) &\geq 0 \\
h(x) &= 0 \\
u^T g(x) &= 0 \\
u &\geq 0.
\end{aligned}
$$

Lagrangian Duality

Definition: A *Lagrangian* is a function of the form

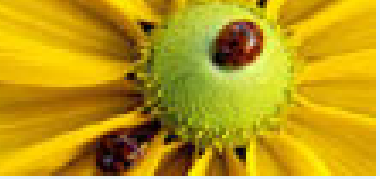
$$L(u,v,x) = \begin{cases} f(x) - u^T g(x) + v^T h(x), & \text{if } u \geq 0, g(x) \geq 0, h(x) = 0 \\ -\infty, & \text{if } u_i < 0, g(x) \geq 0, h(x) = 0 \\ \infty, & \text{otherwise.} \end{cases}$$

Let $\Theta(u,v) = \inf_{x: g(x) \geq 0, h(x)=0} L(u,v,x)$.

Define the dual problem as

$$\begin{array}{ll} \max_{u,v} & \Theta(u,v) \\ \text{s.t.} & u \geq 0. \end{array}$$

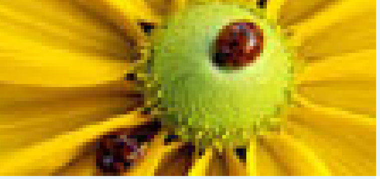
We have weak duality, but not strong.



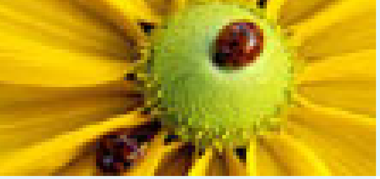
Example: Dual of an NLP

Find the dual problem for

$$\begin{aligned} \text{min} \quad & x^2 \\ \text{s.t.} \quad & x \geq 1. \end{aligned}$$



Methods for Solving Constrained NLPs



Barrier Methods

Consider the inequality constrained NLP:

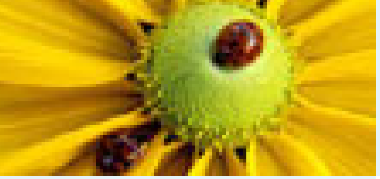
$$\begin{array}{lll} \text{min} & f(x) & \\ \text{s.t.} & g(x) & \geq \quad 0 \end{array}$$

- Approach the solution from the inside of the feasible region.
- Start and stay feasible.
- Two possible choices:

$$\text{min } f(x) - \mu \sum_{i=1}^{m} \log(g_i(x))$$

$$\text{min } f(x) + \mu \sum_{i=1}^{m} \frac{1}{g_i(x)}$$

- Start μ very large and drive it to 0 as $g_i(x)$ go to 0 for some i.



Penalty Methods

Consider the equality constrained NLP:

$$\begin{array}{lccc} \text{min} & f(x) & & \\ \text{s.t.} & h(x) & = & 0 \end{array}$$

- Approach the solution from the outside of the feasible region.
- Two possible choices:

$$\text{min } f(x) + \frac{1}{2}\rho \sum_{i=1}^{m} (g_i(x))^2$$

$$\text{min } f(x) + \rho \sum_{i=1}^{m} |g_i(x)|$$

- Start ρ small and increase it until the optimal solution to the original problem is reached.