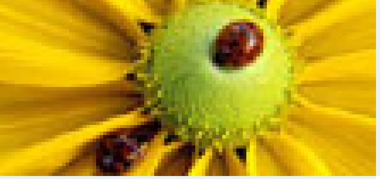
Week 2

Basics of Unconstrained Optimization

OPR 992

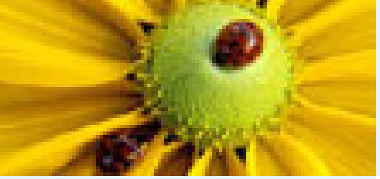
Applied Mathematical Programming



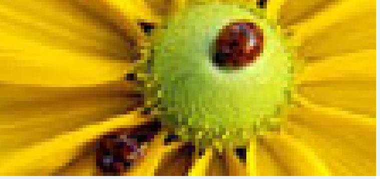
Problem Formulation

$$\min_x f(x)$$

We are interested in finding local minima.



Optimality Conditions



Calculus as usual

- We assume that f is twice continuously differentiable.
- Let x_* be a local minimum.
- First-order necessary condition: $\nabla f(x_*) = 0$.
- Second-order necessary condition: $\nabla^2 f(x_*)$ is positive semidefinite.

Example 1

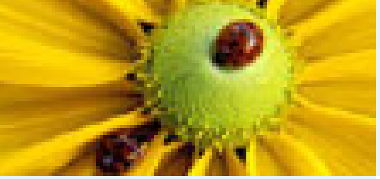
Find all local minima for

$$f(x) = (x_2 - x_1^2)(x_2 - 2x_1^2).$$

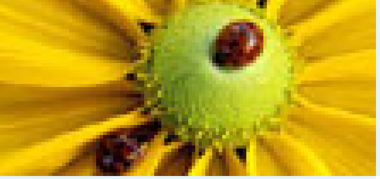
Example 2

Find all local minima for

$$f(x) = 2x_1^2 + x_2^2 - 2x_1x_2 + 2x_1^3 + x_1^4.$$

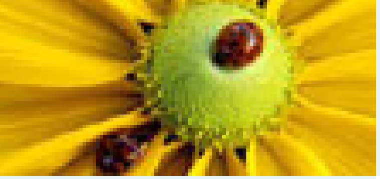


Methods for Solving Unconstrained NLPs



Common Characteristics

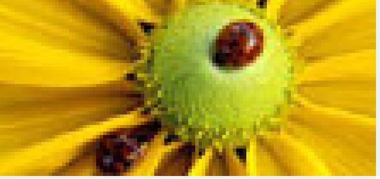
1. Start at x_0 and $k = 0$.
2. Compute step direction Δx_k using some method.
3. Find a steplength α_k such that $x_{k+1} = x_k + \alpha_k \Delta x_k$ gives sufficient descent.
4. Let $x_{k+1} = x_k + \alpha_k \Delta x_k$ and $k = k + 1$.
5. If $\|\nabla f(x_{k+1})\| < \epsilon$, stop. Otherwise, go to step 2.



Finding the right steplength

$$\min f(x_1, x_2) = 5x_1^2 + 7x_2^2 - 3x_1x_2$$

Let $x_k = (2, 3)$ and $\Delta x_k = (-5, -7)$. Choose a steplength that will guarantee descent.



Newton’s Method

- Finds the roots of an equation.
- To find a stationary point, we need $\nabla f(x) = 0$.
- Let
$$\Delta x_k = -[\nabla^2 f(x_k)]^{-1} \nabla f(x_k).$$
- If $\nabla^2 f(x_k)$ is positive semidefinite, it is invertible and Δx_k is a descent direction.
- Fast convergence near the optimum.

Steepest Descent

- No second derivative computations
- No system of equations to solve
- Let

$$\Delta x_k = -\nabla f(x_k).$$

- Simple to implement, but not very good convergence properties.

Quasi-Newton Methods

- Eliminate the burden of computing second derivatives by replacing them with an estimate
- A good estimate should be sparse and positive semidefinite.
- Let

$$\Delta x_k = -B_k^{-1} \nabla f(x_k).$$

- BFGS is one scheme for estimating the second derivative.