Asymptotic Tracking and Robustness of MAS Transitions Under a New Communication Topology

Hossein Rastgoftar, Harry G. Kwatny, Life Fellow, IEEE, and Ella M. Atkins, Senior Member, IEEE

Abstract—We have recently applied the principles of continuum mechanics to develop a new leader–follower model for the collective motion of a multiagent system (MAS). Agents are modeled as particles of a continuum body that can deform in $\mathbb{R}^n$ ($n = 1, 2, 3$) under a specific class of mappings, called the homogeneous transformation. This paper shows how a desired homogeneous deformation is uniquely specified based on the trajectories chosen by $p+1$ ($p \leq n$) leaders, and it is acquired by the remaining agents, called followers, through local communication. Under this setup, every follower interacts with $p+1$ local agents with fixed communication weights that are uniquely determined based on the initial positions of the agents. Although asymptotic convergence of the agents’ transient positions to the desired final positions (prescribed by a homogeneous transformation) can be assured by applying the proposed paradigm, follower agents deviate from the desired positions during evolution. The main objective of this paper is to assure that the transient error, the difference between the actual and desired positions of each follower, converges to zero during evolution. For this purpose, each leader chooses a time-dependent polynomial vector of order $(n-1)$ ($n \in \mathbb{N}$) for its trajectory connecting two consecutive way points, and each follower applies continuous time or discrete time linear time invariant dynamics to update its state based on the states of $p+1$ local agents. The second objective of this paper is to develop a paradigm for the homogeneous deformation of an MAS that is robust to communication failure. For this purpose, we will show how followers can acquire desired positions prescribed by a homogeneous mapping to preserve volumetric ratios under either fixed or switching communication topologies, where there is no restriction on the number of the agents, if every follower communicates with $m_j \geq p+1$ local agents. In addition, agents’ collective motion can be stably continued even if some followers give up communication with other agents at some time during evolution.

Note to Practitioners—A multiagent system evolution as continuum deformation is a novel idea for avoiding interagent collision and collisions of agents with obstacles, while a capability to manage large deformations (expansion or contraction) is provided. This idea becomes more interesting if deviations of agents from desired positions defined by continuum deformation vanish during evolution when agents only access neighboring agent state information.

Index Terms—Asymptotic tracking, collective motion, continuum deformation, multiagent systems (MASs).

I. INTRODUCTION

DISTRIBUTED control for a multiagent system (MAS) has received considerable attention in recent years. Distributed control algorithms have been widely applied for the collective motion of mobile robot systems. Collective motion of an MAS has found different applications, such as gaming, maneuvering in hazardous environments, and formation flights. The consensus algorithm [3]–[17], methods based on the partial differential equation (PDE) [18]–[22], and the containment control method [23]–[28] are recent approaches to managing collective motion of an MAS. These methods are all inspired by heat diffusion problems and apply Laplacian control to reach global coordination through local communication.

To date, different applications, such as motion control [3], [4], network clock synchronization [6], medical [7], [8], and power systems [10], have been proposed for the consensus model. In [11] and [12], it has been shown how an MAS applying switching communication topologies can asymptotically reach consensus or agreement. Furthermore, the robustness of distributed convergence in an MAS under communication failure [16] and model uncertainty [29] are important issues that have been addressed by researchers. Coming to a consensus agreement under stochastically switching topologies was also developed in [30]. Necessary and sufficient conditions for convergence of an MAS under consensus when the communication graph is generated by an ergodic and stationary random process were developed in [17]. Stability analysis of MAS evolution under consensus with time delay has also been studied [13]–[15].

When applying PDE methods, evolution of an MAS is usually modeled by the first-order or second-order PDE with spatially varying parameters. Agents of the MAS are then categorized as leaders and followers, where leader agents are those agents placed at the MAS team boundary. Leaders’ positions are prescribed by the imposed boundary conditions. The interior agents are followers where each follower communicates with some neighboring agents, and the communication weights are determined by the discretization of the spatially varying PDE element(s). Ghods and Krstic [18] and Frihauf and Krstic [20] show how a random distribution of agents can be stably deployed on desired planar curves, where the collective motion of the agents is
governed by a reaction-advection-diffusion class of PDEs. Meurer and Krstic [22] show how a collective motion of an MAS, prescribed by a nonlinear PDE, can be deployed on a desired formation through local communication. In [21], a PDE-based model reference adaptive control algorithm has been proposed for the collective motion of an MAS, facing uncertain heterogeneous interagent communication. A PDE-based technique has been recently applied in wind-integrated power systems to control interarea power oscillations when connected to the transmission line [19].

Containment control is a leader–follower model for prescribing a collective motion of an MAS. Under this setup, leader agents move independently and guide all agents toward a desired configuration. In addition, followers update their positions through local interaction with positive communication weights, where interagent communication among the followers is determined by a strongly connected subgraph. In [28], hybrid Go–Stop control strategies enable leaders to guide followers to a desired target. Containment control of an MAS under both fixed [26] and switching communication topologies is developed in [24], [26], and [27]. Containment control of a formation where agents are modeled as double integrators was presented in [25], and communication weight formulation was investigated in [23]. In [24], control algorithms are designed, such that all followers converge into the convex hull spanned by the leaders while the leaders traverse polygonal trajectories. Containment control can theoretically assure that the agents’ positions asymptotically converge to desired positions inside the convex hull defined by the leaders. Nonetheless, there remain issues to clearly address. For instance, interagent collision is not necessarily avoided during evolution under a fixed interaction topology if communication weights applied by the followers are not consistent with agents positions in the initial formation. In addition, followers may leave the containment region during transit even though they later converge to the convex hull prescribed by the leaders. This may result in interagent collision and collisions of agents with obstacles during transit or evolution.

We have recently applied the principles of continuum mechanics to propose a new leader–follower paradigm for the collective motion of an MAS in \( \mathbb{R}^n \) (\( n = 1, 2, 3 \)) [1], [2]. The fundamentals of homogeneous MAS deformation under no communication and local communication are presented in [2]. In [2], an MAS is treated as particles of a continuum that deform in \( \mathbb{R}^n \) under a homogeneous mapping. Note that homogeneous deformation is a linear and nonsingular mapping with a Jacobian that is time-varying but not spatially varying. Due to linearity, a desired homogeneous transformation in \( \mathbb{R}^n \) can be uniquely related to the trajectories of \( n + 1 \) leaders, where leaders are placed at the vertices of a deformable polytope in \( \mathbb{R}^n \) called a leading polytope. A desired homogeneous transformation of an MAS can be achieved under no communication when positions of the leaders are predefined for the follower agents over a finite horizon of time [2]. In addition, a desired homogeneous deformation can be acquired by every follower agent through local communication with \( n + 1 \) local agents, where: 1) interagent communication among the followers is prescribed by a connected graph and 2) communication weights are all consistent with the initial positions of agents [2]. In [1], we specified an upper limit for deviation of each follower from the desired position prescribed by a homogeneous transformation, where each follower only accesses the positions of its in-neighbor agents but the desired position is globally specified based on the trajectories chosen by \( n + 1 \) leaders. We showed that this upper limit depends on the maximum velocity of the leaders during evolution, total number of agents, dimension of the motion space, control gain applied by each follower, and the norm of the inverse of the communication matrix. In addition, the polyhedral communication topology proposed in [1] can be applied by follower agents to acquire a desired homogeneous deformation through preservation of some volumetric (area) ratios specified based on the initial positions of the agents.

Continuum deformation of an MAS, which has some similarity with the available containment control method, can assure that interagent collision is avoided while the MAS has the capability of large deformation. Furthermore, it can be guaranteed that followers remain inside the transient convex hull defined by the leaders’ positions at any time. Two issues not addressed in the previous work include vanishing followers’ deviations from their desired positions during transition, and robustness to communication failure. Therefore, this paper has two main objectives. First, this paper shows that the transient error, i.e., the difference between actual and desired follower positions, vanishes during MAS evolution. Followers are, therefore, shown not to exit the transient convex hull defined by the leaders’ positions at any time \( t \). For this purpose, we first let each leader choose a finite polynomial of order \( (h - 1) \in \mathbb{N} \) for its trajectory, connecting two consecutive way points. Dynamics of order \( h \in \mathbb{N} \) describes the evolution of each follower and show how deviations of followers from the desired positions converge to zero during transition while each follower only accesses the state information of \( n + 1 \) local agents. The second contribution of this paper is to assure that MAS evolution remains robust even during communication failure. In this regard, this paper applies the method of preservation of volumetric ratios developed in [1]. Without loss of generality, MAS evolution in a plane (\( \in \mathbb{R}^2, n = 2 \)) is considered, and thus, each follower is allowed to communicate with \( m_i \geq n + 1 = 3 \) local agents. Followers acquire desired positions by preserving certain area ratios that are determined based on the initial positions of the follower \( i \) and set of \( m_i \) agents adjacent to \( i \). Because followers are not limited to communicate with only three local agents, there is no restriction on the total number of agents of the MAS, and MAS evolution is robust to communication failure. Hence, if some followers give up communication with other agents at some time (during MAS evolution), a new communication topology can be applied by the remaining followers, and homogeneous deformation through local communication continues.

Asymptotic tracking of desired positions prescribed by a homogeneous deformation must be distinguished from the trajectory tracking problem. In a trajectory tracking problem, desired displacements of the followers are the same, because follower trajectories are prescribed by one or more leaders with the same displacements, indicating a rigid transformation.
For example, the trajectory chosen by a leader moving with constant magnitude acceleration can be asymptotically tracked by followers if they apply the second-order consensus algorithm and update their accelerations based on both positions and velocities of in-neighbor agents [24], [31]. In addition, deviations of followers from the desired trajectories, prescribed by three nonaligned leaders having the same displacements, can converge to zero while leaders move if followers apply the second-order consensus dynamics [24]. Under a homogeneous deformation, on the other hand, interagent distances can be expanded or contracted to support deformation. The volume occupied by the MAS can be contracted if passing through a narrow channel is required, where deviations of followers from the desired positions can vanish during deformation.

This paper is organized as follows. We first summarize pertinent graph theory and homogeneous deformation concepts in Sections II and III. Section IV presents a formal problem statement. In Sections V and VI, the asymptotic tracking of desired positions and the interagent collision avoidance are mathematically proved. Continuum deformation of an MAS using a polyhedral communication topology is discussed in Section VII. Simulation results are shown in Section VIII followed by concluding remarks in Section IX.

II. Preliminaries

A. Graph Theory Notions

Let $G = (V, E)$ be a directed graph that prescribes interagent communication of an MAS with nodes $V = \{1, 2, \ldots, N\}$ and edges $E \subset V \times V$. A node $i$ can access state information of node $j$ if $(j, i) \in E$. The in-neighbor set $N_i = \{j : (j, i) \in E\}$ defines the agents whose states are accessible to the node $i$, where $|N_i|$ denotes the cardinality of $N_i$. Let graph $G$ include boundary graph $\partial \Phi$ and subgraph $\Phi$, where $\Phi$ is the node set of subgraph $\Phi$. Then, the nodes belonging to the boundary $\partial \Phi$ are defined as

$$V_{\partial \Phi} = \{j \in V \setminus V_{\Phi} : \forall i \in V_{\Phi}: (j, i) \in E\}.$$  

Two nodes $i$ and $j$ in directed graph $G$ are connected if there is a directed path from $i$ to $j$ and a directed path from $j$ to $i$. A directed graph $G$ is weakly connected if substituting directed edges by undirected edges yields a connected graph. A directed graph is strongly connected if every two graph nodes $i$ and $j$ are connected. A vertex or node set $V$ includes a leader set

$$V_L = \{i \in V : |N_i| = 0\}$$

and a follower set

$$V_F = V \setminus V_L.$$  

In other words, every node $i$ of communication graph $G$ represents either a leader agent that moves independently, or a follower agent, which updates its position based on the positions of the agents belonging to the in-neighbor set $N_i$.

Definition 1: A node $k$ of communication graph $G$ is called a leader if $|N_k| = 0$. Leader agents are defined by a leader set

$$V_L = \{k : |N_k| = 0\}.$$  

Leaders are identified by $1, 2, \ldots, N_L$, thus

$$V_L = \{1, 2, \ldots, N_L\}.$$  

Definition 2: A node $i$ of communication graph $G$ is called a follower if it belongs to a follower set $V_F = V \setminus V_L$. Followers are identified by $N_L + 1, N_L + 2, \ldots, N$, thus

$$V_F = \{N_L + 1, N_L + 2, \ldots, N\}.$$  

B. Geometric Notions

Let

$$r = \sum_{l=1}^{n} x_l \hat{e}_l$$

denote linear space $M \subset \mathbb{R}^n$ position with orthonormal unit basis $(\hat{e}_1, \ldots, \hat{e}_n)$. $M$ can be rewritten as an indirect sum

$$M = M_p \oplus M_q$$

where $M_p \subset \mathbb{R}^p$ and $M_q \subset \mathbb{R}^{n-p}$ are the linear subspaces of $M$ with orthonormal unit basis $(\hat{e}_1, \ldots, \hat{e}_p)$ and $(\hat{e}_{p+1}, \ldots, \hat{e}_n)$. For each $r \in M$, there exists unique $r_p \in M_p$ and $r_q \in M_q$, such that $r = r + r_q$, where

$$r_p = \sum_{l=1}^{p} x_l \hat{e}_l$$

and

$$r_q = \sum_{l=p+1}^{n} x_l \hat{e}_l.$$  

1) Evolution Space: Agents of an MAS evolve in a linear space $M \subset \mathbb{R}^n$, called the evolution space, where agent position is given by

$$r_i = \sum_{q=1}^{n} x_{q,i}(t) \hat{e}_q.$$  

2) Deformation Subspace: Let the evolution of agents in $\mathbb{R}^n$ be guided by $p + 1 \ (p \leq n)$ leaders located at the vertices of a transient $p$-dimensional convex polytope belonging to the linear subspace $M_p \subset M$ with unit basis $(\hat{e}_1, \ldots, \hat{e}_p)$. The linear subspace $M_p$ is called the deformation subspace.

III. Homogeneous Deformation of an MAS

A continuum (deformable body) is a continuous region in $\mathbb{R}^n$, containing an infinite number of infinitesimal particles. Initial positions of material particles of a continuum are called material coordinates and denoted by $R = \tilde{R} \oplus R_q \ (R \in \mathbb{R}^n)$. Let the deformation of a continuum in the linear subspace $M_p$ be defined by a mapping $\tilde{r}(R, t) \in \mathbb{R}^p$, where

- The Jacobian $Q$ is nonsingular. Note that $Q(\tilde{R}, t) = (\partial \tilde{r}(\tilde{R}, t)/\partial R) \in \mathbb{R}^{p \times p}$.
- $\tilde{r}(\tilde{R}, t_0) = \tilde{R} \ (t_0$ denotes the initial time).

Because $Q(\tilde{R}, t)$ is nonsingular, no two particles of the continuum occupy the same position during deformation. This interesting property can guarantee the avoidance of the interagent collision if MAS evolution is treated as continuum deformation. A continuum deformation is called homogeneous if the Jacobian matrix $Q$ is only a function of...
time \( Q = Q(t) \). Homogeneous deformation of an MAS is, therefore, given by
\[
\tilde{r}_i(t) = Q(t) \tilde{R}_i + D(t), \quad i \in V \tag{2}
\]
where
\[
\tilde{R}_i = \sum_{l=1}^{p} \tilde{X}_{i,l} \hat{e}_l = \sum_{l=1}^{n} X_{i,l} \hat{e}_l
\]
and
\[
\tilde{r}_i = \sum_{l=1}^{p} \tilde{x}_{i,l} \hat{e}_l = \sum_{l=1}^{n} x_{i,l} \hat{e}_l
\]
are the initial and current positions of the agent \( i \), respectively, and \( D \in \mathbb{R}^n \) is a rigid body displacement vector.

A. Homogeneous Deformation of the Leading Polytope

Let \( \tilde{r}_1(t), \tilde{r}_2(t), \ldots, \tilde{r}_{p+1}(t) \) be the positions of \( N_l = p + 1 \) leaders at the vertices of a \( p \)-dimensional leading convex polytope evolving in linear subspace \( M_p \subset M \subset \mathbb{R}^n \). Then
\[
\forall t \geq t_0, \quad \text{Rank}[\tilde{r}_2 - \tilde{r}_1 \ldots \tilde{r}_{p+1} - \tilde{r}_1] = \text{Rank}[\tilde{r}_2 - r_1 \ldots r_{p+1} - r_1] = p \tag{3}
\]
and the position of the follower \( i (i \in V_F) \) at a time \( t \) can be uniquely expressed as follows:
\[
\tilde{r}_i(t) = \tilde{r}_1(t) + \sum_{k=2}^{p+1} a_{i,k}(t)(\tilde{r}_k(t) - \tilde{r}_1(t))
\]
where \( a_{i,1}(t) = 1 - (a_{i,2} + a_{i,3} + \ldots + a_{i,p+1}) \) or
\[
\sum_{k=2}^{p+1} a_{i,k}(t) = 1. \tag{5}
\]
Let \( \tilde{r}_j(t) (j \in V \) spanning the index number of leader \( k \in V_L \) or follower \( i \in V_F \) in (4) be replaced by
\[
\tilde{r}_j(t) = \sum_{l=1}^{p} \tilde{x}_{i,j}(t) \hat{e}_l, \quad j \in V \tag{6}
\]
where the values of \( \hat{e}_1, \hat{e}_2, \ldots, \hat{e}_p \) form the orthogonal unit basis of the Cartesian coordinate system, and \( x_{i,j}(t) \) is the \( j \)-th, \( l = 1, 2, \ldots, p \) agent \( j \in V \) position component in deformation subspace \( M_p \). Then, (4) becomes
\[
\sum_{l=1}^{p+1} \tilde{x}_{i,l}(t) \hat{e}_l = \sum_{k=1}^{p} a_{i,k}(t) x_{i,k}(t) \hat{e}_l. \tag{7}
\]
By considering (5) and (7), parameter \( a_{i,k}(t) \) \((i \in V_F \) and \( k \in V_L)\) is uniquely determined at any time \( t \). Under a homogeneous deformation, parameter \( a_{i,k}(t) \) \((i \in V_F \) and \( k \in V_L)\) remains time invariant and is denoted by \( a_{i,k} \). Thus, the desired position of the follower \( i \in V_F \) in deformation subspace \( M_p \) becomes
\[
\tilde{r}_{i,HT}(t) = \sum_{k=1}^{p+1} a_{i,k} \tilde{r}_k(t) \tag{8}
\]
where \( a_{i,k} \) is uniquely determined based on the initial positions of agent \( i \) and the \( p + 1 \) leaders by
\[
\begin{bmatrix}
\tilde{X}_{1,1} & \tilde{X}_{1,2} & \ldots & \tilde{X}_{1,p+1} \\
\tilde{X}_{2,1} & \tilde{X}_{2,2} & \ldots & \tilde{X}_{2,p+1} \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{X}_{p,1} & \tilde{X}_{p,2} & \ldots & \tilde{X}_{p,p+1}
\end{bmatrix}
\begin{bmatrix}
\tilde{a}_{1,1} \\
\tilde{a}_{1,2} \\
\vdots \\
\tilde{a}_{i,p} \\
\tilde{a}_{i,p+1}
\end{bmatrix}
= \begin{bmatrix}
\tilde{H}_{1,1} \\
\tilde{H}_{2,1} \\
\vdots \\
\tilde{H}_{p,1} \\
1
\end{bmatrix}. \tag{9}
\]

The desired position of follower \( i \in V_F \) (in evolution space) is given by
\[
r_{i,HT}(t) = \sum_{k=1}^{p+1} a_{i,k} r_k(t). \tag{10}
\]

Note that the entries of Jacobian matrix \( Q(t) \) and vector \( D(t) \) can be uniquely determined, based on the components of the leaders’ positions [2], by
\[
J_l = [I_p \otimes L_0 \quad I_p \otimes 1]^{-1} P_l \tag{11}
\]
where \( 1 \in \mathbb{R}^{p+1} \) is the one vector, \( I_p \in \mathbb{R}^{p \times p} \) is the identity matrix, “\( \otimes \)" is the Kronecker product symbol
\[
J_l = [Q_{11} \ldots Q_{pp} \quad D_1 \ldots D_p]^T \in \mathbb{R}^{(p+1)p}
\]
and
\[
P_l = [\tilde{x}_{1,1} \ldots \tilde{x}_{1,p+1} \ldots \tilde{x}_{p,1} \ldots \tilde{x}_{p,p+1}]^T \in \mathbb{R}^{(p+1)p}
\]
B. 2-D Deformation

Suppose \( R_i = \tilde{X}_i \hat{e}_x + \tilde{Y}_i \hat{e}_y \) and \( r_i(t) = \tilde{x}_i(t) \hat{e}_x + \tilde{y}_i(t) \hat{e}_y \), denote the initial and current positions of agent \( i \in V \) expressed in the Cartesian coordinate system with unit basis \( (\hat{e}_x, \hat{e}_y) \). Homogeneous deformation of a continuum in the \( \tilde{X}\tilde{Y} \) plane is given by
\[
\begin{bmatrix}
\tilde{x}_i(t) \\
\tilde{y}_i(t)
\end{bmatrix} = \begin{bmatrix}
Q_{11}(t) & Q_{12}(t) \\
Q_{21}(t) & Q_{22}(t)
\end{bmatrix}
\begin{bmatrix}
\tilde{X}_i(t) \\
\tilde{Y}_i(t)
\end{bmatrix} + \begin{bmatrix}
D_1(t) \\
D_2(t)
\end{bmatrix}. \tag{12}
\]
If the leader agents remain nonaligned at any time \( t \), then the entries of Jacobian matrix \( Q \in \mathbb{R}^{2 \times 2} \) and the vector \( D \in \mathbb{R}^2 \) can be related to the \( \tilde{X} \) and \( \tilde{Y} \) components of the leaders’ positions as follows [1], [2]:
\[
\begin{bmatrix}
Q_{11}(t) \\
Q_{12}(t) \\
Q_{21}(t) \\
Q_{22}(t)
\end{bmatrix} = \begin{bmatrix}
\tilde{X}_1(t) & \tilde{Y}_1(t) & 0 & 0 & 1 & 0 \\
\tilde{X}_2(t) & \tilde{Y}_2(t) & 0 & 0 & 1 & 0 \\
\tilde{X}_3(t) & \tilde{Y}_3(t) & 0 & 0 & 1 & 0 \\
0 & 0 & \tilde{X}_1(t) & \tilde{Y}_1(t) & 0 & 1 \\
0 & 0 & \tilde{X}_2(t) & \tilde{Y}_2(t) & 0 & 1 \\
0 & 0 & \tilde{X}_3(t) & \tilde{Y}_3(t) & 0 & 1
\end{bmatrix}^{-1}
\begin{bmatrix}
\tilde{x}_1(t) \\
\tilde{y}_1(t) \\
\tilde{x}_2(t) \\
\tilde{y}_2(t) \\
\tilde{x}_3(t) \\
\tilde{y}_3(t)
\end{bmatrix}. \tag{13}
\]
Furthermore, the desired position \( r_{i,HT}(t) \) specified by the homogeneous deformation becomes
\[
r_{i,HT} = a_{i,1} r_1(t) + a_{i,2} r_2(t) + a_{i,3} r_3(t) \tag{14}
\]
where time invariant parameters $a_{i,1}$, $a_{i,2}$, and $a_{i,3}$ are obtained from

$$
\begin{bmatrix}
\bar{X}_1 & \bar{X}_2 & \bar{X}_3 \\
\bar{Y}_1 & \bar{Y}_2 & \bar{Y}_3
\end{bmatrix} \begin{bmatrix}
a_{i,1} \\
a_{i,2} \\
a_{i,3}
\end{bmatrix} = \begin{bmatrix}
\hat{X}_i \\
\hat{Y}_i
\end{bmatrix}.
$$

Sections IV and V will show how each follower $i \in V_F$ can acquire desired position $r_{1,HT}(t)$ through local communication, where a transient error of follower $i$ vanishes during MAS evolution.

IV. PROBLEM STATEMENT AND OBJECTIVES

A. Motivation

Containment control can be applied to theoretically ensure the asymptotic convergence of MAS evolution to a final formation when all communication weights are arbitrarily chosen to be positive [32]. However, implementing containment control does not necessarily assure interagent collision avoidance. For example, consider an MAS with six agents, three leaders, and three followers, with an initial formation and communication topology, as shown in Fig. 1. Let leader agents be placed at the vertices of a leading triangle and let them horizontally move according to

$$
r_k(t) = \tilde{r}_k(t) = R_k + \frac{1}{3} t \hat{e}_x, \quad k = 1, 2, 3
$$

where $R_k = \tilde{R}_k = X_k \hat{e}_x + Y_k \hat{e}_y$, and $r_k(t) = x_k(t) \hat{e}_x + y_k(t) \hat{e}_y$ are the initial and current positions of leader $k$ ($k = 1, 2, 3$), respectively. In addition, let followers update their positions according to

$$
\dot{r}_i = g \sum_{j \in N_i} w_{i,j} (r_j - r_i)
$$

where $g = 2$ and the weights of communication $w_{4,1} = 1/3$, $w_{4,5} = 1/3$, $w_{4,6} = 1/3$, $w_{5,2} = 1/3$, $w_{5,4} = 1/3$, $w_{5,6} = 1/3$, $w_{6,3} = 0.5$, $w_{6,4} = 0.25$, $w_{6,5} = 0.25$ are all positive. Since leader agents move horizontally, the $Y$ coordinates of all followers remain invariant during evolution, so the $Y$ position components of followers 4 and 5 remain identical during MAS evolution (see Fig. 1). Shown in Fig. 2(a) are the $X$ components of actual positions of the followers 4 and 5 versus time. As shown in Fig. 2(a), these two agents collide at $t = 0.45 s$ as the $Y$ components of their initial positions are identical. All containment control protocol requirements were satisfied yet collision occurred. This is because the communication weights of the followers are not consistent with the initial MAS formation.

This issue can be managed by treating an MAS as particles of a continuum where transient desired positions are prescribed by a nonsingular deformation mapping. Due to nonsingularity of the continuum deformation, desired positions of no two agents are the same during evolution while interagent distances can be expanded or contracted. Collision is thus assured, and the MAS has the ability to pass through a narrow passage. Without loss of generality, consider a homogeneous deformation in $\mathbb{R}^n$ ($n = 1, 2, 3$) uniquely obtained by the trajectories of $p + 1 (p \leq n)$ leaders located at the vertices of a convex polytope. Followers acquire the transient desired homogeneous mapping prescribed by the leaders’ positions through local communication. For this purpose, we apply containment control, such that a homogeneous deformation is prescribed by the independent evolution of $p + 1$ leaders. Followers locally communicate with weights that are consistent with the initial positions of their in-neighbor agents.

B. Problem Statement and Objectives

Consider an MAS with $N$ agents that moves in $\mathbb{R}^n$ ($n = 1, 2, 3$) under the containment control protocol. The MAS contains $p + 1$ leaders ($V_L = \{1, 2, \ldots, p+1\}$) located at the vertices of the leading polytope. The remaining $N - p - 1$ followers ($V_F = \{p + 2, \ldots, N\}$) are distributed inside the leading polytope at initial time $t_0$.

Agent $i$’s position is updated by

$$
\frac{d^h r_i}{dt^n} = u_i
$$

where

$$
u_i = \begin{cases} 0, & i \in V_L \\ \sum_{k=1}^{h} \beta_k \frac{d^{h-k}}{dt^{h-k}} (r_{i,d} - r_i), & i \in V_F. \end{cases}
$$

Note that

$$r_{i,d} = \sum_{j \in N_i} w_{i,j} r_j$$

is the local desired position for the follower $i$ and $w_{i,j} \in \mathbb{R}_+$ is the communication weight of the follower $i \in V_F$. In addition, $\beta_k \in \mathbb{R}_+$ ($k = 1, 2, \ldots, p$) is a constant control gain. The parameters $\beta_k$ must be chosen, such that the roots of the agent dynamics characteristic equations are all located in the open left half $s$-plane. It is noticed that the leader $i \in V_L$ chooses the polynomial vectors of order $p - 1$ for its trajectory connecting two consecutive way points.
To model MAS evolution as continuum deformation, followers, which are initially placed inside the leading polytope, choose communication weights that are consistent with agents’ initial positions. The characteristic equations for the followers’ communication weights were given in [1] and [2] and will be obtained in Section IV-A of this paper. Rastgoftar and Jayasuriya [1], [2] show how followers can acquire final desired positions prescribed by a homogeneous deformation after leaders stop. However, followers deviate from the desired positions during transition because of single integrator dynamics. If this deviation is large, the followers may leave the containment region. For example, consider the initial, intermediate, and final formations of an MAS shown in Fig. 3. The MAS consists of six agents (three leaders and three followers) and applies the suggested first-order leader–follower model in [1] and [2] to evolve in the plane ($\in \mathbb{R}^2$). While the follower agents ultimately reach the desired final positions if leaders and followers update their positions according to the dynamics in (18) and (19).

MAS homogeneous deformation is achieved through preserving some volumetric ratios specified based on initial agent positions. Homogeneous deformation under the preservation of volumetric ratios is advantageous, because followers are allowed to interact with more than $n + 1$ local agents, where fixed and switching communication topologies can be applied by followers to acquire their desired positions through local communication.

V. ASYMPTOTIC TRACKING OF DESIRED POSITIONS

A. Communication Topology

Let interagent communication be prescribed by graph $G$ where $p + 1$ nodes belonging to $\partial \Phi$ represent leaders and the remaining $(N - p - 1)$ nodes belonging to directed, strongly connected subgraph $\Phi$ represent followers. Notice that every follower $i$ ($i \in V_F$) communicates with $p + 1$ local agents ($|N_i| = p + 1$). It is assumed that each leader moves independently but its position is tracked by a follower. Hence, leader–follower communication is unidirectional and shown by an arrow terminated at a follower. A typical MAS communication graph to move in a plane ($\in \mathbb{R}^2$) is shown in Fig. 4.

B. Communication Weights and Weight Matrix

Let follower $i$ ($i \in V_F$) and $p + 1$ adjacent agents belonging to an in-neighbor set $N_i = \{i_1, i_2, \ldots, i_{p+1}\}$ be initially placed at $R_{i_1}, R_{i_2}, R_{i_3}, \ldots, R_{i_{p+1}}$, where

$$\text{Rank}[R_{i_2} - R_{i_1} \ldots R_{i_{p+1}} - R_{i_1}] = \text{Rank}[\tilde{R}_{i_2} - \tilde{R}_{i_1} \ldots \tilde{R}_{i_{p+1}} - \tilde{R}_{i_1}] = p. \quad (21)$$

Then, the initial position of follower $i$ can be uniquely expanded as a linear combination of the initial positions of the $p + 1$ adjacent agents by

$$R_i = \sum_{k=1}^{p+1} w_{i,k} R_{i_k} \quad \quad (22)$$

where

$$\sum_{k=1}^{p+1} w_{i,k} = 1.$$ 

Communication weight $w_{i,k}$ is uniquely determined by solving the following set of $p + 1$ linear algebraic equations:

$$[\hat{X}_{1,i_1} \; \hat{X}_{1,i_2} \; \ldots \; \hat{X}_{1,i_{p+1}} \; w_{i_1,i} \; w_{i_2,i} \; \ldots \; w_{i_{p+1},i}] = [\tilde{X}_{1,i} \; \tilde{X}_{2,i} \; \ldots \; \tilde{X}_{p,i}] \quad \quad (23)$$

It is assumed that each follower $i$ ($\forall i \in V_F$) is initially placed inside the convex communication polytope whose vertices are occupied by the $p + 1$ in-neighbor agents. Therefore, communication weights are all positive. Communication weights for the initial distribution shown in Fig. 4 are given by (23) and listed in Table II.

Weight matrix $W \in \mathbb{R}^{(N-p-1) \times N}$ with entry $W_{ij}$ is given by

$$W_{ij} = \begin{cases} w_{i,j+p+1} > 0, & \text{if } (j, i + p + 1) \in E \\ -1, & \text{if } i + p + 1 = j \\ 0, & \text{otherwise.} \end{cases} \quad (24)$$
Let weight matrix $W$ be partitioned as follows:

\[ W = [B \in \mathbb{R}^{(N-p-1)\times(p+1)} \quad A \in \mathbb{R}^{(N-p-1)\times(N-p-1)}]. \tag{25} \]

The matrix $A$ is Hurwitz as shown in the Appendix.

**Remark 1:** Let

\[ \hat{U}_{l,q} = [\hat{x}_{l,1} \ldots \hat{x}_{l,p+1}]^T \in \mathbb{R}^{p+1} \]

and

\[ \hat{Z}_{l,q} = [\hat{x}_{l,p+2} \ldots \hat{x}_{l,N}]^T \in \mathbb{R}^{N-p-1} \]

denote the $l$th components of the initial positions of the leaders and the followers, respectively. Then

\[ A\hat{Z}_{l,1} + B\hat{U}_{l,1} = 0. \tag{26} \]

**C. Difference Between Communication Weight and Communication Quality**

In this paper, communication weights are based on inter-agent distances determined from initial positions. They do not represent the quality (strength) of a communication link between two connected nodes. As will be demonstrated in Theorem 1, interagent communication can be specified by a directed graph. This directed graph can define interagent communication links, i.e., the in-neighbor agent(s) for each follower, based on link quality or strength as well as energy efficiency [33]. Each follower then updates its reference position based on the positions of its in-neighbor agents with the weights that are consistent with the positions of agents.

**Theorem 1:** Let MAS leader and follower initial positions be given. If the positions of the $p + 1$ leaders satisfy the rank condition (3), then the matrix

\[ W_L = -A^{-1}B = \begin{bmatrix} a_{p+2,1} & \cdots & a_{p+2,p+1} \\ \vdots & \ddots & \vdots \\ a_{N,1} & \cdots & a_{N,p+1} \end{bmatrix} \]  

is invariant for any arbitrary graph $G = (V, E)$ with the directed and strongly connected subgraph $\Phi = (V_F, E_F)$ ($E_F \in V_F \times V_F$), where $|N_i| = p + 1$ ($i \in V_F$), the positions of the agents belonging to the in-neighbor set $N_i$ satisfy the rank condition (21), and the communication weights are defined by (23).

**Proof:** Since leaders’ positions satisfy rank condition (3), $\hat{x}_{l,i}$, the $l$th ($l = 1, 2, \ldots, p$) component of an initial follower $i$ position, can be uniquely expressed as a linear combination of the $l$th leader position components as follows:

\[ \hat{x}_{l,i} = \sum_{k=1}^{p+1} a_{i,k} \hat{x}_{l,k}, \quad i \in V_F, k \in V_L \]

where $a_{i,k}$ is uniquely determined by applying (9). Let

\[ \tilde{U}_{l,1} = [\tilde{x}_{l,1} \ldots \tilde{x}_{l,p+1}]^T \in \mathbb{R}^{p+1} \]  

and

\[ \tilde{Z}_{l,1} = [\tilde{x}_{l,p+2} \ldots \tilde{x}_{l,N}]^T \in \mathbb{R}^{N-p-1} \]

denote the $l$th components of the initial positions of the leaders and followers, respectively. Then, $\tilde{Z}_{l,1}$ and $\tilde{U}_{l,1}$ are related by

\[ \tilde{Z}_{l,1} = W_L \tilde{U}_{l,1}. \tag{29} \]

Note that row $i$ of (29) denotes the $l$th component of the initial follower $i + p + 1 \in V_F$ position expressed as a linear combination of the $l$th components of the positions of the leaders according to (28).

On the other hand, $\hat{x}_{l,i}$ can be expressed as a linear combination of $\hat{x}_{l,i}$, $\hat{x}_{l,i+1}$, $\ldots$, $\hat{x}_{l,p+1}$, where the initial positions of the adjacent agents satisfy rank condition (21). Therefore, (23) has unique solutions for the communication weights of a follower agent $i$ ($\forall i \in V_F$). By considering (24) and (25), it is concluded that

\[ W \begin{bmatrix} \hat{U}_{l,1} \\ \hat{Z}_{l,1} \end{bmatrix} = A\hat{Z}_{l,1} + B\hat{U}_{l,1} = 0. \tag{30} \]

Since subgraph $\Phi$ is strongly connected and communication weights are positive, matrix $A$ is Hurwitz [2]. Therefore

\[ \tilde{Z}_{l,1} = -A^{-1}B\tilde{U}_{l,1}. \tag{31} \]

By equating the right-hand sides of (29) and (31), it is concluded that $W_L = -A^{-1}B$.

**Corollary:** The $l$th components of the followers’ desired positions prescribed by a homogeneous transformation of the initial configuration can be defined as

\[ \tilde{Z}_{l,HT}(t) = W_L\tilde{U}_{l}(t) = \begin{bmatrix} a_{p+2,1} & \cdots & a_{p+2,p+1} \\ \vdots & \ddots & \vdots \\ a_{N,1} & \cdots & a_{N,p+1} \end{bmatrix} \begin{bmatrix} \hat{x}_{l,1} \\ \vdots \\ \hat{x}_{l,p+1} \end{bmatrix} \]

\[ \hat{x}_{l,i,HT}(t) = \sum_{k=1}^{p+1} a_{i,k} \hat{x}_{l,k}(t) \]

where $\hat{x}_{l,i,HT}(t) = [\hat{x}_{l,i+1,HT}(t) \ldots \hat{x}_{l,N,HT}(t)]^T \in \mathbb{R}^{N-p-1}$, and $\tilde{Z}_{l,HT}(t)$ is the $l$th component of desired position $r_{i,HT}(t)$ ($i \in V_F$) given by (10).

**D. Dynamics of Agents**

1) Continuous Time Domain: Let

\[ r_i = \sum_{l=1}^{n} x_{l,i} \hat{e}_i, \quad i \in V \]

expressed with respect to the Cartesian coordinate system with unit basis ($\hat{e}_1, \ldots, \hat{e}_n$) be updated by (18) and (19), then $x_{i,HT}(t)$, the $l$th component of the position of the follower $i$, is updated by the following dynamics:

\[ \frac{d^h x_{l,i}}{dt^h} = \sum_{k=1}^{h} \beta_k \frac{d^{h-k}}{dt^{h-k}} \left( \sum_{j \in N_i} w_{i,j} x_{i,j} - x_{l,i} \right). \tag{34} \]

Equation (34) is row $i - p - 1$ of the following matrix dynamics of order $h$:

\[ \frac{d^h Z_l}{dt^h} = A \sum_{k=1}^{h} \beta_k \frac{d^{h-k}}{dt^{h-k}} Z_l = B \sum_{k=1}^{h} \beta_k \frac{d^{h-k}}{dt^{h-k}} U_l \]

\[ U_l = [x_{l,1} \ldots x_{l,p+1}]^T \in \mathbb{R}^{p+1} \]

and

\[ Z_l = [x_{l,p+2} \ldots x_{l,N}]^T \in \mathbb{R}^{N-p-1} \]
denote the $l$th leader and follower position components, respectively.

Notice that the MAS evolution dynamics given by (35) is asymptotically stable if the characteristic polynomial

$$|s^h I - (\beta_1 s^{h-1} + \cdots + \beta_{h-1} s + \beta_h) A| = 0$$

is Hurwitz.

**Remark 2:** Given $A$ and $B$ specified by (24) and (25), the $l$th components of the transient desired positions of the followers in $\mathbb{R}^n$ are obtained by

$$Z_{l,HT} = W_L U_L(t)$$

where

$$Z_{l,HT} = [x_{l,p+2,HT}, \ldots, x_{l,N,HT}]^T \in \mathbb{R}^{N-p-1}$$

and

$$x_{l,i,HT}(t) = \sum_{k=1}^{p+1} a_{l,k} x_{l,k}(t), \quad i \in V_F.$$ 

**Transient Error Dynamics:** The right-hand side of (35) can be rewritten as

$$B \sum_{k=1}^{h} \beta_k \frac{d^{h-k}}{dt^{h-k}} U_l = A A^{-1} B \sum_{k=1}^{h} \beta_k \frac{d^{h-k}}{dt^{h-k}} U_l$$

$$= -A \sum_{k=1}^{h} \beta_k \frac{d^{h-k}}{dt^{h-k}} W_L U_l$$

$$= -A \sum_{k=1}^{h} \beta_k \frac{d^{h-k}}{dt^{h-k}} Z_{l,HT}.$$ 

Thus, (35) is converted to

$$\frac{d^h E_l}{dt^h} - A \sum_{k=1}^{h} \beta_k \frac{d^{h-k}}{dt^{h-k}} E_l = \frac{d^h Z_{l,HT}}{dt^h}$$

where $E_l = Z_{l,HT} - Z_l$ is the $l$th component of the difference between the global desired position and the actual position of the follower. Because each leader trajectory is defined by a finite order polynomial vector of order $(h - 1)$, $\frac{d^h Z_{l,HT}}{dt^h} = 0$ and $E_l$ asymptotically converge to zero. In other words, deviations of followers from the desired states prescribed by homogeneous mapping (32) vanish during MAS evolution while followers only interact with local agents.

Next, the discrete time dynamics for follower evolution is presented.

2) **Discrete Time Domain:** Recall that the leader $i \in V_L$ trajectory is defined by a polynomial vector of order $(h - 1) \in \mathbb{N}$. Therefore, $\frac{d^h r_l}{dt^h} (i \in V_L)$ vanishes, and we can assume that $x_{l,i}[K]$ satisfies the following difference equation:

$$\sum_{k=0}^{h} \sigma_k x_{l,i}[K + 1 - k] = 0$$

where

$$\sigma_k = \binom{h}{k} \frac{(-1)^k}{\Delta t^h}$$

and $\Delta t$ is the time increment. It is further assumed that the position of the follower $i \in V_F$ is updated according to the following difference equation of order $h$:

$$x_{l,i}[K + 1] = \sum_{k=0}^{h} (-\sigma_{k+1} x_{l,i}[K] - k) + \gamma_k \sum_{j \in N_i} w_{i,j} (x_{l,j}[K - k] - x_{l,i}[K - k])$$

where the values of $\gamma_0$, $\gamma_1$, $\ldots$, $\gamma_{h-1}$ are constant scalar parameters.

**Collective Dynamics:** Equation (42) is row $i - p - 1$ of the following matrix difference equation of order $h$:

$$Z_i[K + 1] = -\sum_{k=0}^{h-1} \sigma_{k+1} Z_i[K - k]$$

$$+ \sum_{k=0}^{h} \gamma_k (AZ_i[K - k] + BU_i[K - k]).$$

The dynamics of (43) is asymptotically stable if all the roots of the characteristic equation

$$|z^h + \left(\sum_{k=0}^{h-1} \sigma_{k+1} I - \gamma_k A\right) z^{h-k-1}| = 0$$

are inside the open disk with radius 1 with center located at the origin. Next, rewrite (43) as follows:

$$\sum_{k=0}^{h} \sigma_k Z_i[K + 1 - k]$$

$$- A \sum_{k=0}^{h} \gamma_k (Z_i[K - k] - Z_{l,HT}[K - k]) = 0$$

where $Z_{l,HT}[K - k] = -A^{-1} BU[K - k]$. By considering (40), it is concluded that

$$\sum_{k=0}^{h} \sigma_k U_i[K + 1 - k] = 0.$$ 

Hence

$$\sum_{k=0}^{h} \sigma_k Z_{l,HT}[K + 1 - k] = -\sum_{k=0}^{h} \sigma_k A^{-1} BU_i[K + 1 - k] = 0$$

and (45) can be rewritten as follows:

$$Z_i[K + 1] - Z_{l,HT}[K + 1]$$

$$+ \sum_{k=0}^{h-1} (\sigma_k I - \gamma_k A) (Z_i[K - k] - Z_{l,HT}[K - k])$$

$$= E_i[K + 1] + \sum_{k=0}^{h-1} (\sigma_k I - \gamma_k A) E_i[K - k] = 0.$$ 

Characteristic equation is the same as (44), which is stable. Therefore, transient error $E_l = Z_l - Z_{l,HT}$, and the difference between the $l$th components of the actual and desired positions of the followers converges to zero during MAS evolution.
VI. COLLISION AVOIDANCE

In this section, each follower is considered to be a ball in \( \mathbb{R}^n \) with radius \( \epsilon > 0 \). Our objective is to specify sufficient conditions to assure the avoidance of interagent collision and collision of agents with obstacles during MAS evolution.

A. Constraint on Leaders’ Motion

1) Constraint 1: Leaders must choose trajectories that satisfy rank condition (3) at each time \( t \) during MAS evolution.

2) Constraint 2: Leaders must choose trajectories, such that the leading polytope does not collide with obstacles in the motion field.

3) Constraint 3: The \( p + 1 \) leaders must choose their trajectories, such that the desired positions of any two followers never get closer than \( 2(\delta + \epsilon) \)

\[
i, j \in V \land i \neq j, \quad ||r_i,HT(t) - r_j,HT(t)|| \geq 2(\delta + \epsilon). \quad (49)
\]

Note that

\[
\delta \geq ||r_i(t) - r_{i,HT}(t)||
\]

is considered the upper limit for the deviation of every follower agent from desired position \( r_{i,HT}(t) \) given by a homogeneous deformation, when each follower acquires homogeneous deformation through local communication. Furthermore, distance of each follower from the boundaries of the leading polytope must never become less than \( (\delta + \epsilon) \).

Theorem 2: Suppose \( m_1 \in V \) and \( m_2 \in V \) are the indices of two agents with minimum separation distance

\[
\gamma _l = ||R_{m_1} - R_{m_2}|| \quad (51)
\]

in the initial formation. Let \( \gamma _s \) denote the minimum distance of a follower from a boundary of the leading polytope in the initial configuration. Graphical representations of \( \gamma _l \) and \( \gamma _s \) for a typical initial distribution in the \( X - Y \) plane are illustrated in Fig. 6. Let

\[
\delta _{max} = \min \left\{ \frac{1}{2}(\gamma _l - 2\epsilon), (\gamma _s - \epsilon) \right\}. \quad (52)
\]

then,

\[
\left( \frac{2(\delta + \epsilon)}{2(\delta _{max} + \epsilon)} \right)^2 = \frac{(\delta + \epsilon)^2}{(\delta _{max} + \epsilon)^2} \leq 1
\]

is the lower limit for the magnitude of eigenvalues of Jacobian matrix \( Q^T Q \in \mathbb{R}^{p \times p} \).

Proof: By using polar decomposition, \( Q(t) = R_c(t)U_D(t) \), where \( R_c \) is orthogonal and \( U_D \) is symmetric. Therefore, \( U_D^2 = Q^T Q \) and

\[
\lambda(U_D) = \sqrt{\lambda(Q^T Q)}.
\]

Because eigenvalues of \( Q \) all have positive real part, \( U_D \) has positive eigenvalues. Note that \( U_D \) is the pure stretch deformation matrix [34], thus, the minimum eigenvalue of the matrix \( U_D \) should not become less than \( (\delta + \epsilon)/(\delta _{max} + \epsilon) \). As described in Section III-A, entries of \( Q \) and \( D \) can be uniquely related to the components of leader positions with Eq. (11) if leaders’ positions satisfy the rank condition (3). Therefore, it is required that leaders choose their trajectories such that the minimum value of the smallest eigenvalues of matrix \( \sqrt{Q^T Q} \) never becomes less than

\[
\lambda_{\min}(\sqrt{Q^T Q}) = \frac{(\delta + \epsilon)}{(\delta _{max} + \epsilon) + \epsilon} \quad (53)
\]

We will specify the upper limit \( \delta \) for follower deviation from the homogeneous transformation with the first- and second-order dynamics as follows.

B. Follower Evolution

1) First-Order Dynamics: Let follower position agent \( i \in V_F \) be updated by the following first-order dynamics:

\[
\dot{i}_l = g(r_{i,d} - r_i). \quad (54)
\]

Then, the \( l \)th position components of the followers are updated by the following first-order dynamics:

\[
\frac{dZ_l}{dt} = g(AZ_l + BU_l) = gA(Z_l - Z_{l,HT}) = gAE_l. \quad (55)
\]

Transient error dynamics \( E_l = Z_{l,HT} - Z_l \) becomes

\[
\frac{dE_l}{dt} - gAE_l = \frac{dZ_l}{dt} \quad (56)
\]

and

\[
E_l(t) = e^{gAt}E_l(t_0) + \int_{t_0}^{t} e^{g(A(t-t)}}\dot{Z}_{l,HT}dt. \quad (57)
\]

Because communication weights are consistent with the agents’ initial positions

\[
AZ_l(t_0) + BU_l(t_0) = 0. \quad (58)
\]

Equation (58) implies that \( Z_{l,HT} = Z_l(t_0) = -A^{-1}BU_l(t_0) \), and consequently, \( E_l(t_0) = Z_{l,HT}(t_0) - Z_l(t_0) \) in the right-hand side of (57) vanishes. Hence

\[
||E_l(t)|| = \int_{t_0}^{t} e^{g(A(t-t)}}||\dot{Z}_{l,HT}||. \quad (59)
\]

Here, it is assumed that the leaders are not necessarily stationary but the maximum of the \( l \)th components of the leader velocities do not exceed upper limit \( V_l \) at any time \( t \) during MAS evolution. Therefore

\[
\left\| \frac{dZ_{l,HT}}{dt} \right\| = \left\| W_L \frac{dU_l}{dt} \right\| \leq V_l ||W_L 1|| \quad (60)
\]
where $1 \in \mathbb{R}^{p+1}$ is one vector, and the positive matrix $W_L \in \mathbb{R}^{(N-p-1) \times (p+1)}$ has rows that sum to 1. This implies
\[
||W_L 1|| = ||1|| = \sqrt{N - p - 1}
\]
and
\[
||\dot{Z}_{i,HT}|| \leq V_t \sqrt{N - p - 1}.
\]
By considering inequality (60), it is concluded that
\[
||\dot{x}_i(t) - x_i,HT(t)|| \leq ||E_i|| \leq \delta_i
\]
where
\[
\delta_i = \frac{||A^{-1}||\sqrt{N - p - 1}}{g} V_t.
\]
Let $V = (\sum_{i=1}^{p} V_i^2)^{1/2}$ be the maximum magnitude for the leaders’ velocities at any time $t$ during MAS evolution. Then
\[
\delta = \frac{||A^{-1}||\sqrt{N - p - 1}}{g} V
\]
specifies an upper bound for the deviation of each follower from the desired state defined by a homogeneous transformation. In other words
\[
||\dot{r}_i(t) - r_i,HT(t)|| \leq \delta = \frac{||A^{-1}||\sqrt{N - p - 1}}{g} V.
\]

2) Higher Order Dynamics: As shown in Section IV, if agent $i \in V$ position is updated by the higher order dynamics given by (18) and (19), then transient error $E_i$ is updated by
\[
d^h \dot{E}_i = A \sum_{k=1}^{h} \beta_k \frac{d^{h-k} \dot{E}_i}{dt^{h-k}} = 0.
\]
Let $K E_i = [E_i^T \ldots (d^{h-1}E_i^T/dt^{h-1})]^T \in \mathbb{R}^{h(N-p-1)}$, then
\[
K E_i = A_E K E_i
\]
where
\[
A_E = \begin{bmatrix}
0_{N-p-1} & I_{N-p-1} & \ldots & 0_{N-p-1} & 0_{N-p-1} \\
0_{N-p-1} & 0_{N-p-1} & \ldots & 0_{N-p-1} & 0_{N-p-1} \\
0_{N-p-1} & 0_{N-p-1} & \ldots & 0_{N-p-1} & 0_{N-p-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0_{N-p-1} & 0_{N-p-1} & \ldots & 0_{N-p-1} & I_{N-p-1} \\
\beta_1 A & \beta_{h-1} A & \ldots & \beta_2 A & \beta_1 A
\end{bmatrix}
\]
(69)

Note that $I_{N-p-1} \in \mathbb{R}^{(N-p-1) \times (N-p-1)}$ and $0_{N-p-1} \in \mathbb{R}^{(N-p-1) \times (N-p-1)}$ are the identity and zero matrices, respectively. In addition, $A_E \in \mathbb{R}^{h(N-p-1) \times h(N-p-1)}$ is Hurwitz. Therefore
\[
K E_i(t) = e^{A_E(t-t_0)} K E_i(t_0)
\]
is the solution of the first-order dynamics in (68). Notice that
\[
||K E_i(t)|| \leq ||K E_i(t_0)||
\]
is less than $\delta = ||K E_q(t_0)||$ at any time $t$ during MAS evolution. Hence, if leaders move in such a way that the inequality (71) is satisfied, the avoidance of interagent collision is guaranteed.

Remark 3: Parameters $\beta_1, \ldots, \beta_k$ should be chosen, such that all the eigenvalues of the matrix $A_E$ are located in the open left half $s$-plane to ensure the stability of the MAS homogeneous deformation.

Remark 4: $E_i(t_0) = Z_i,HT(t_0) - Z_i(t_0)$, the first partition of the vector $K_i(t_0) \in \mathbb{R}^{h(N-p-1)}$, but the remaining partitions $[(dE_i(t_0)/dt), \ldots, (d^{h-1}E_i(t_0)/dt^{h-1})]$ are not necessarily zero. If $K_i(t_0) = 0$ is a zero vector, then the transient deviation of each follower from desired state $||\dot{r}_i(t) - r_{i,HT}(t)||$, $\forall i \in V_F$ vanishes during MAS evolution, while followers only access the positions of their in-neighbor agents. In other words, homogeneous transformation is perfectly tracked due to zero deviations of the followers from the state of homogeneous deformation if
\[
d^k Z_i(t_0) = \frac{d^k Z_i,HT(t_0)}{dt^k} = W_L \frac{d^k U_i(t_0)}{dt^k}, \quad k = 1, 2, \ldots, h-1.
\]

VII. AREA PRESERVATION METHOD

In this section, we introduce an alternative method for achieving homogeneous MAS deformation under local communication, where each follower $i$ is allowed to interact with $m_i \geq p+1$ local agents. Therefore, the continuum deformation of an MAS under switching topologies can be assured, even if some of the followers do not communicate with other agents during MAS evolution. In addition, the convergence rate of the MAS evolution can be enhanced as followers are permitted to increase communications.

Let $G = \{G_1, G_2, \ldots, G_n\}$ be the set of interagent communication graphs applied by the followers to evolve in $\mathbb{R}^n$. It is assumed that the graph $G_k = \Phi_k \oplus \partial \Phi_k$ specifies interagent communication for any graph $G_k$ of the form $G_k(t) = \{\{t_{i-1}, t_i\}, \ldots, \{t_{n-1}, t_n\}\}$, where the values of $t_0, t_1, \ldots, t_n$ are the switching times, connected subgraph $\Phi_k$ defines communication among the followers, and $|N_k(t)| \geq p + 1$ for every node $i$ belongs to $\Phi_k$. This implies that the followers are permitted to interact with more than $m_i(t) \geq p + 1$ local agents. Notice that the nodes belonging to $\partial \Phi_k$ representing leader agents remain unchanged but the total number of the nodes and edges of subgraph $\Phi_k$ can change when switching occurs. A recently developed follower evolution model based on the preservation of volumetric ratios [1] is considered in the continuation of this section. We still desire that the collective motion of the agents is prescribed by a homogeneous mapping. Without loss of generality, we consider an MAS evolution problem in a plane ($e \in \mathbb{R}^2$). MAS evolution is guided by three leaders placed at the vertices of a leading triangle, while followers are inside the leading triangle. Follower $i \in V_F$ can interact with $m_i(t) \geq 3$ local agents and acquire desired positions by preserving area ratios that are consistent with the agents positions.
A. Assumption

For MAS evolution under area preservation, it is assumed that the evolution and deformation spaces are the same ($M = M_p$).

B. Communication Topology

Let graph $G_k$ specify interagent communication for all $t \in [t_{k-1}, t_k]$, where a follower agent $i$ can access to the positions of adjacent agents $i_{1,k}, i_{2,k}, \ldots, i_{m_i,k}$. Consider a communication polygon $\Omega_{i,k}$ for the follower agent $i$ whose vertices are occupied by agents $i_{1,k}, i_{2,k}, \ldots, i_{m_i,k}$. Fig. 6 shows communication polygon $\Omega_{i,k}$ considered as the union of $m_i$ triangles, where the agents $i, i_{j,k}, i_{j+1,k}$ are placed at the vertices of the $j$th ($j = 1, 2, \ldots, m_i-1$) triangle.

C. Area Weights

Consider communication polygon $\Omega_{i,k}$ with net area $ar_{i,k}(t)$ at time $t \in [t_{k-1}, t_k]$ ($k = 1, 2, \ldots$). Let $r_i = x_i\hat{e}_x + y_i\hat{e}_y$, $r_j = x_j\hat{e}_x + y_j\hat{e}_y$, and $r_{j+1} = x_{j+1}\hat{e}_x + y_{j+1}\hat{e}_y$ be the positions of the follower $i$ and the adjacent agents $i_{j,k}$ and $i_{j+1,k}$, where $\hat{e}_x$ and $\hat{e}_y$ are the orthogonal unit bases for the $xy$ plane. The area of the $j$th triangle inside communication polygon $\Omega_{i,k}$ at time $t \in [t_{k-1}, t_k]$ is given by

$$ar_{i,j,k} = 1/2 \begin{vmatrix} x_i & y_i & 1 \\ x_{i,j,k} & y_{i,j,k} & 1 \\ x_{i,j+1,k} & y_{i,j+1,k} & 1 \end{vmatrix} = O_{i,j,k}x_i + P_{i,j,k}y_i + Q_{i,j,k}. \quad (72)$$

The area of the polygon $\Omega_{i,k}$ does not depend on the position of the follower agent $i$, therefore

$$\sum_{j=1}^{m_i} O_{i,j,k} = \sum_{j=1}^{m_i} P_{i,j,k} = 0 \quad (73)$$

$$ar_{i,k}(t) = \sum_{j=1}^{m_i} Q_{i,j,k}. \quad (74)$$

Let $AR_{i,j,k} = ar_{i,j,k}(t_{k-1})$ and $AR_{i,k} = ar_{i,k}(t_{k-1})$, then

$$AW_{i,j,k} = \frac{AR_{i,j,k}}{AR_{i,k}}, \quad i \in V_F, \quad j \in N_i \quad (75)$$

is the $j$th area weight for follower $i$ during $t \in [t_{k-1}, t_k]$.

D. Cost of Homogeneous Deformation

Consider

$$J_{i,k}(t) = \frac{1}{2} \sum_{j=1}^{m_i} (ar_{i,j,k}(t) - AW_{i,j,k}ar_{i,k}(t))^2 \quad (76)$$

as the cost imposed on the follower $i$ to update its current position, such that the transient area weight

$$aw_{i,j,k}(t) = \frac{ar_{i,j,k}}{ar_{i,k}} \quad (77)$$

remains as close as possible to area weights $AW_{i,j,k}$. Note that the cost $J_{i,k}(t)$ ($\forall i \in V_F$) vanishes at all times $t \in [t_{k-1}, t_k]$, if agents’ positions at time $t \in [t_{k-1}, t_k]$ satisfy the homogeneous transformation.

E. Desired Follower Positions

Let cost function $J_{i,k}(t)$ be locally minimized at $r_{i,k} = x_{i,k}\hat{e}_x + y_{i,k}\hat{e}_y$. Then, by satisfying $(\partial J_{i,k}/\partial x_i) = 0$ and $(\partial J_{i,k}/\partial y_i) = 0$, the $x$ and $y$ components of $r_{i,k}$ are obtained as follows:

$$\begin{bmatrix} x_{i,k} \\ y_{i,k} \end{bmatrix} = -\begin{bmatrix} \sum_{j=1}^{m_i} O_{i,j,k}^2 & \sum_{j=1}^{m_i} O_{i,j,k}P_{i,j,k} \\ \sum_{j=1}^{m_i} O_{i,j,k}P_{i,j,k} & \sum_{j=1}^{m_i} P_{i,j,k}^2 \end{bmatrix}^{-1} \begin{bmatrix} \sum_{j=1}^{m_i} O_{i,j,k}Q_{i,j,k} - \sum_{q=1}^{m_i} \sum_{j=1}^{m_i} AW_{i,j,k}O_{i,j,k}Q_{i,j,k} \\ \sum_{j=1}^{m_i} O_{i,j,k}Q_{i,j,k} - \sum_{q=1}^{m_i} \sum_{j=1}^{m_i} AW_{i,j,k}O_{i,j,k}Q_{i,j,k} \end{bmatrix}. \quad (78)$$

F. MAS Evolution Dynamics

It is assumed that follower $i \in V_F$ updates its position according to

$$\dot{r}_i = g(r_{id,k} - r_i) \quad (79)$$

where $r_i$ and $r_{id,k}$ are the actual and desired positions of the agents $i$ in evolution space, respectively. Equilibrium states denoted by $r_{id,k}(t)$ are locally stable, because the velocity of follower $i$ is directed toward equilibrium state $[r_{id,k}(t)]$ at any time $t \in [t_{k-1}, t_k]$, where level curves $J_{i,k} = \text{constant}$ are all convex.1 Therefore, $J_{i,k} \leq 0$ in the vicinity of $r_{id,k}$ and cost function $J_{i,k}(t)$ remain bounded. Consequently, the total cost for homogeneous mapping

$$J_{k}(t) = \sum_{i=0}^{N} J_{i,k}(t) \quad (80)$$

remains bounded, and the MAS evolution dynamics is locally stable. In the last switch ($t \in [t_{n-1}, t_n]$) where interagent communication is prescribed by $G_{n}$, area weight $AW_{i,j,n}$ ($i \in V_F, j \in V$) is calculated based on the positions of

1The level curves $J_{i,k} = \text{constant}$ are ellipses centered at the origin (Please see the Fig. 8).
the agents at time $t = t_{n-1}$, and $J_{i,n-1}$ asymptotically tends to zero when the leaders stop. Therefore, the final MAS formation is a homogeneous transformation of the agent configurations at time $t = t_{n-1}$.

VIII. SIMULATION RESULTS

In this section, we present the results of simulating a 1-D homogeneous deformation in a 2-D evolution space and a 2-D homogeneous deformation in a 3-D evolution space. For the 1-D homogeneous transformation, the deformation subspace $M_p$ is moving. Fig. 7 shows the deformation subspace $M_p$ with unit vector $\tilde{e}_x$ rotating in the $X - Y$ plane. For the 2-D homogeneous transformation, the deformation subspace is stationary, thus unit vectors $\tilde{e}_x$ and $\tilde{e}_y$ are fixed.

A. 1-D Homogeneous Deformation in a 2-D Space

Consider an MAS consisting of six agents (two leaders and four followers) evolving in the $X - Y$ plane. As shown in Fig. 7, agents are initially placed along leading line segments defined by the leaders’ positions. Initial positions of the leaders are listed in Table I. The unit vector

$$\tilde{e}_i(0) = \frac{(X_2 - X_1)\tilde{e}_x + (Y_2 - Y_1)\tilde{e}_y}{\sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2}}$$

is determined at the initial time $0$. Then, $\tilde{X}_i = R_t \cdot \tilde{e}_i(0)$ is obtained as listed in Table I, and communication weights are determined by applying (23). Note that for the 1-D homogeneous deformation, (23) simplifies to

$$w_{i,i_1} = \frac{\tilde{X}_{i_2} - \tilde{X}_i}{\tilde{X}_{i_2} - \tilde{X}_{i_1}}$$

$$w_{i,i_2} = \frac{\tilde{X}_i - \tilde{X}_{i_1}}{\tilde{X}_{i_2} - \tilde{X}_{i_1}}. \quad (81)$$

Follower communication weights are listed in the last two columns of Table I.

![Fig. 7. Schematic of the 1-D homogeneous deformation in the $xy$ plane.](image1)

![Fig. 8. Level curves $J_{i,k} = $ constant.](image2)

![Fig. 9. Leader paths in examples 1 and 2.](image3)

Table I

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![VIII. Simulation Results](image4)

**Table I**

**Follower Communication Weights in Scenario I**

- Leader 1: $\tilde{X}_1(0) = (0, 0, 0)$, $\tilde{Y}_1(0) = (0, 0, 0)$
- Leader 2: $\tilde{X}_2(0) = (10, 0, 0)$, $\tilde{Y}_2(0) = (10, 0, 0)$
- Follower 1: $\tilde{X}_3(0) = (5, 0, 0)$, $\tilde{Y}_3(0) = (5, 0, 0)$
- Follower 2: $\tilde{X}_4(0) = (0, 5, 0)$, $\tilde{Y}_4(0) = (0, 5, 0)$
- Follower 3: $\tilde{X}_5(0) = (0, 0, 5)$, $\tilde{Y}_5(0) = (0, 0, 5)$
- Follower 4: $\tilde{X}_6(0) = (5, 5, 0)$, $\tilde{Y}_6(0) = (5, 5, 0)$

$\delta_{max} = (\gamma_1/2) - \epsilon = 0.1850m$. We consider four different scenarios for collective motion of the MAS in the $\tilde{X} - \tilde{Y}$ plane. In the first two examples leaders negotiate a narrow channel and move with constant velocities on straight paths shown in Fig. 9. The $\tilde{X}$ and $\tilde{Y}$ components of the leaders positions are given by

\[
\begin{align*}
\tilde{x}_1(t) &= \frac{8}{T}t - 5 \quad 0 \leq t \leq T \\
\tilde{y}_1(t) &= \frac{2}{T}t - 4 \quad 0 \leq t \leq T \\
\tilde{x}_2(t) &= \frac{10}{T}t \quad 0 \leq t \leq T \\
\tilde{y}_2(t) &= \frac{1}{T}t + 1 \quad 0 \leq t \leq T \\
\tilde{x}_3(t) &= \frac{6}{T}t - 3 \quad 0 \leq t \leq T \\
\tilde{y}_3(t) &= \frac{3}{T}t + 5 \quad 0 \leq t \leq T
\end{align*}
\]

where $T$ denotes the time when the leaders arrive at their final destinations inside the narrow channel.

Entries of Jacobian $Q \in \mathbb{R}^{2 \times 2}$ and vector $D \in \mathbb{R}^{2 \times 2}$ of the desired homogeneous transformation are obtained based on the $\tilde{X}$ and $\tilde{Y}$ components of the leaders positions are
shown versus $t/T$ in Fig. 10. Note that $Q$ and $D$ define homogeneous deformation in the $X - Y$ plane. Furthermore, eigenvalues of $\sqrt{Q^TQ}$ are depicted versus $\frac{t}{T}$ in Fig. 11. As shown, $\lambda_{\text{min}}(\sqrt{Q^TQ}) = 0.4322$ is the minimum eigenvalue of matrix $\sqrt{Q^TQ}$.

In the first and second examples shown above, followers apply first and second order linear models to update their own positions. In the third and fourth examples, leaders choose second order polynomial vectors for their positions. The third example shows how each follower can asymptotically track desired positions $r_{\alpha,HT}$ by applying stable third order discrete time dynamics to access positions of in-neighbor agents. Robustness to communication failure is demonstrated in example 4 where follower agents evolve under the area preservation technique.

**Example 1 (First Order Continuous Time Dynamics):** Given $\epsilon = 0.025$ m, $\lambda_{\text{min}}(\sqrt{Q^TQ}) = 0.4322$ and $\delta_{\text{max}} = 0.1850$m, the upper bound for deviation of the followers is obtained from Eq. (53) as follows:

$$
\delta = \frac{\lambda_{\text{min}}(\sqrt{Q^TQ})}{T}(\delta_{\text{max}} + \epsilon) - \epsilon = 0.0657m.
$$

It is noted that $V = \frac{\sqrt{10^2 + (-1)^2}}{T}$ is the upper limit of the leader velocities when they move according to the trajectories given in Eq. (84). Let follower $i \in V_F$ ($V_F = \{4, 5, \ldots, 14\}$) update its current position by applying the first order dynamics given in Eq. (54) where inter-agent communication is defined by the graph shown in Fig. 1 and communication weights listed in Table I are applied. Then, matrices $A \in \mathbb{R}^{11 \times 11}$ and $B \in \mathbb{R}^{11 \times 9}$ are obtained by using definitions (24) and (25). Given $||A^{-1}|| = 7.8917$, $\delta = 0.0657$m, $V = ((10^2 + (-1)^2)^{1/2}/T)$, $N = 14$, and $p = 2$ in Eq. (66),

$$
gT \geq \frac{7.8917 \times \sqrt{11} \times \sqrt{101}}{0.0657} = 4.0037 \times 10^3.
$$

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Selecting $T = 65$ and $g = 65$, $gT \geq 4.0037 \times 10^3$ assures that deviation of each follower necessarily remains less than upper limit $\delta$ during MAS evolution $(\forall i \in V_F, ||r_i(t) - r_{\alpha,HT}(t)|| \leq \delta = 0.0657m)$.

In Figs. 12(a) and 12(b) the $\tilde{X}$ and $\tilde{Y}$ components of desired position $r_{14,HT}(t)$ and actual position $r_{14}(t)$ are shown by dotted and continuous trends, respectively. As shown, follower 14 ultimately reaches the position specified by the homogeneous transformation,

$$
t \geq 40, r_{14,HT} = a_{14,1}r_1 + a_{14,2}r_2 + a_{14,3}r_3 = 0.2846(3\tilde{e}_x - 2\tilde{e}_y) + 0.3352(10\tilde{e}_x + 3\tilde{e}_y) = 5.3467\tilde{e}_x + 0.1914\tilde{e}_y.
$$

In Fig. 13, $\tilde{x}_{14} - \tilde{x}_{14,HT}$, $\tilde{y}_{14} - \tilde{y}_{14,HT}$, and $||r_{14} - r_{14,HT}||$ are depicted versus time. Note that deviation of follower 14 is less than $\delta = 0.0657$.
and dotted curves in Fig. 15(b). In addition, and desired positions of agent 1 are shown by continuous Fig. 14. MAS formations at different sample times. RASTGOFTAR et al. according to the double-integrator kinematic model followers apply the communication graph in Fig. 4 and the Suppose leaders choose trajectories defined by (82), and the MAS reaches the desired formation inside the narrow passage. Example 2 (Second-Order Continuous Time Dynamics): For this example, leader agents choose the following second-order polynomial vectors for their trajectories: 

\begin{align}
\text{Leader 1:} & \quad \tilde{x}_1(t) = 0.0022t^2 + t - 5, \quad 0 \leq t \leq 60 \\
& \quad \tilde{y}_1(t) = 0.006t^2 - 4, \quad 0 \leq t \leq 60 \\
\text{Leader 2:} & \quad \tilde{x}_2(t) = 0.0028t^2 \leq t \leq 60 \\
& \quad \tilde{y}_2(t) = -0.0003t^2 + 1, \quad 0 \leq t \leq 60 \\
\text{Leader 3:} & \quad \tilde{x}_3(t) = 0.0017t^2 - 3, \quad 0 \leq t \leq 60 \\
& \quad \tilde{y}_3(t) = -0.0008t^2 + 5, \quad 0 \leq t \leq 60. \tag{84}
\end{align}

Followers use the communication graph shown in Fig. 4 throughout evolution, where communication weights are the same as the weights in examples 1 and 2 listed in Table II. Here, every follower agent updates its position by 

\begin{align}
\dot{r}_i &= \beta_1(\dot{r}_{\text{d},i} - \dot{r}_i) + \beta_2(r_{\text{d},i} - r_i). \tag{83}
\end{align}

Then, the characteristic polynomial of the collective dynamics (36) is stable if \( \beta_1 = \beta_2 = 20 \) and \( r_{\text{d},i} \) is determined in Section IV-B. Note that the roots of characteristic polynomial (36) are the same as the eigenvalues of matrix \( A_E \) defined by (69).

In Fig. 15(a), the \( \tilde{X} \) components of the follower 14 actual position and its desired position prescribed by a homogeneous transformation are shown by continuous and dotted curves, respectively. In addition, the \( \tilde{Y} \) components of the actual and desired positions of agent 14 are shown by continuous and dotted curves in Fig. 15(b). In addition, \( ||\tilde{r}_{14} - r_{14,\text{HT}}|| \), the deviation of follower 14 from the desired state, is shown versus time in Fig. 16. Note that the deviation of follower 14 converges to zero during MAS evolution around \( t = 3 \) s.

In Fig. 17, the evolution of the MAS in the \( X-Y-Z \) space is shown. The MAS deforms in the \( \tilde{X} - \tilde{Y} \) plane where \( \tilde{X} - \tilde{Y} \) is shown in Fig. 17 (green mesh).

Example 3 (Discrete Time Kinematic Model): For this example, leader agents choose the following second-order polynomial vectors for their trajectories:

\begin{align}
\text{Leader 1:} & \quad \bar{x}_1(t) = 0.0022t^2 + t - 5, \quad 0 \leq t \leq 60 \\
& \quad \bar{y}_1(t) = 0.006t^2 - 4, \quad 0 \leq t \leq 60 \\
\text{Leader 2:} & \quad \bar{x}_2(t) = 0.0028t^2 \leq t \leq 60 \\
& \quad \bar{y}_2(t) = -0.0003t^2 + 1, \quad 0 \leq t \leq 60 \\
\text{Leader 3:} & \quad \bar{x}_3(t) = 0.0017t^2 - 3, \quad 0 \leq t \leq 60 \\
& \quad \bar{y}_3(t) = -0.0008t^2 + 5, \quad 0 \leq t \leq 60. \tag{84}
\end{align}

Followers use the communication graph shown in Fig. 4 throughout evolution, where communication weights are the same as the weights in examples 1 and 2 listed in Table II. Here, every follower agent updates its position by 

\begin{align}
1.20600r_{14}[K + 1] - 0.20600r_{14,d}[K + 1] - 3.40601r_{14}[K] \\
+ 0.40601r_{14,d}[K + 1] - 3.20000r_{14}[K - 1] - 0.20000r_{14,d}[K - 1] \\
- 1.00000r_{14,d}[K - 2] = 0 \tag{85}
\end{align}

at time step \( K \) where \( \Delta t = 0.01 \). Fig. 18 shows the \( \tilde{X} \) and \( \tilde{Y} \) components of \( r_{14}(t) - r_{14,\text{HT}}(t) \). The \( \tilde{X} \) and \( \tilde{Y} \) components of the desired and actual positions are approximately the same after \( t = 20 \) s, implying the deviation of follower 14 converges to zero, and followers can asymptotically achieve their desired positions.

Example 4 (MAS Evolution Under Area Preservation): Let leaders move on straight paths shown in Fig. 9 where followers apply area preservation to evolve in the \( \tilde{X} - \tilde{Y} \) plane.
Assume that the interagent communication graph $G_1$ shown in Fig. 19(a) is used by the followers to update their positions for all $t \in [0, 10]$ s. However, the follower agent 14 does not communicate with the MAS at $t \geq 10$ s. The new agent communication graph $G_2$ [Fig. 19(b)] is applied by the followers to acquire the desired positions after $t \geq 10$ s.

Area weights $AW_{i,j,1}$ and $AW_{i,j,2}$ ($i \in V_F, j \in V$) corresponding to communication graphs $G_1$ and $G_2$ are listed in Tables III and IV, respectively. Here, every follower updates its position using (79) where $g = 50$. In Fig. 20, the $\tilde{x}$ and $\tilde{y}$ components of the actual and desired positions of follower agent 13 are shown versus time. The MAS formations at $t = 0$ s, $t = 10$ s, and $t = 30$ s are shown in Fig. 21.

IX. CONCLUSION

In this paper, we considered a communication-based framework for the evolution of an MAS under a homogeneous mapping. We admitted the higher order dynamics for the evolution of every follower agent. Asymptotic tracking of a desired homogeneous mapping under a fixed communication topology was guaranteed. We developed an area preservation technique as a tool for addressing robustness to communication failure. This technique can be used to achieve the homogeneous transformation of the MAS under a fixed communication topology where: i) there is no restriction on the number of agents and ii) every follower agent $i$ can communicate with $m_i \geq 3$ local agents. We showed how MAS evolution can remain robust to communication failure if some followers fail to communicate with other agents at some point during evolution. Future work is required to consider the problem of MAS continuum deformation in the presence of heterogeneous communication delay where each follower applies the higher order dynamics to asymptotically track the desired position given by a homogeneous deformation.

APPENDIX

**Theorem 3:** If subgraph $\Phi$ defining interaction among followers is directed and strongly connected, and communication weights are all positive, then partition $A$ of $W$ is Hurwitz.

**Proof:** The sum of each of the first $p + 1$ rows of $A = -(I - F)$ is negative, while the sum of each of the remaining rows of $A$ is zero, because the sum of every row...
of $W$ is zero and $A$ is obtained by eliminating the first $p + 1$ columns of $W$. The matrix $F$ is nonnegative, since all communication weights are positive. In addition, $F$ is irreducible, since the subgraph $\Phi$ is connected. Consequently, from the Perron–Frobenius theorem, it is concluded that the spectral radius of $F$, $\rho(F)$, is not $> 1$. Since the sum of the first rows of $A$ is negative, $A$ cannot be singular, and thus $\rho(F) \neq 1$. As a result, the spectrum of $F$ is $< 1$, and the matrix $A$ is Hurwitz.

ACKNOWLEDGMENT

The authors gratefully acknowledge M. Stevens and S. Balachandran for their help in preparing the multimedia contents and plots.

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Harry G. Kwatny (M’70–SM’82–F’97–LF’06) received the B.S.M.E. degree from Drexel University, Philadelphia, PA, USA, the M.S. degree in aeronautics and astronautics from the Massachusetts Institute of Technology, Cambridge, MA, USA, and the Ph.D. degree in electrical engineering from the University of Pennsylvania, Philadelphia, PA, USA. He is currently a member of the Mechanical Engineering Mechanics Department, Drexel University, where he is also the S. Herbert Raynes Professor of Mechanical Engineering. His current research interests include modeling, analysis and control of nonlinear, parameter-dependent systems with applications to electric generating plants, power systems, aircraft, symbolic computing as a vehicle for bringing advances in nonlinear dynamics and control theory into engineering practice, and the control of impaired systems, in particular, aircraft and power systems.

Prof. Kwatny was the Founding Chairman of the IEEE Automatic Control Society Technical Committee on Energy and Power Systems and a member of the IFAC Committee on Power Plants and Power Systems. He has been an Associate Editor of the IEEE TRANSACTIONS OF AUTOMATIC CONTROL and the IFAC journal Automatica.

Ella M. Atkins (M’07–SM’12) is currently an Associate Professor with the Department of Aerospace Engineering, University of Michigan, Ann Arbor, MI, USA, where she is also the Director of the Autonomous Aerospace Systems Laboratory. She has authored over 150 refereed journal and conference publications. Her current research interests include task and motion planning, guidance, and control to support increasingly autonomous cyber-physical aerospace systems with a focus on small unmanned aircraft system and aviation safety applications.

Dr. Atkins is an AIAA Associate Fellow, a Small Public Airport Owner/Operator with Shamrock Field Airport, Brooklyn, MI, USA, and a Private Pilot. She has served long-term as an Associate Editor of the AIAA Journal of Aerospace Information Systems. She has served on numerous review boards and panels, including the 2013 NRC Committee to develop a research agenda for autonomy in civil aviation. She was the Chair of the AIAA Intelligent Systems Technical Committee. She served on the National Academy’s Aeronautics and Space Engineering Board from 2011 to 2015, and was a member of the IDA Defense Science Studies Group from 2012 to 2013. She serves on the Steering Committee and as Graduate Program Chair to the new University of Michigan Robotics Program.