INTRODUCTION TO OPTIMAL CONTROL SYSTEMS

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OUTLINE

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What is Optimal Control?

- Optimal control is an approach to control systems design that seeks the best possible control with respect to a performance metric.
- The theory of optimal control began to develop in the WW II years. The main result of this period was the Wiener-Kolmogorov theory that addresses linear SISO systems with Gaussian noise.
- A more general theory began to emerge in the 1950’s and 60’s
  - In 1957 Bellman published his book on Dynamic Programming
  - In 1960 Kalman published his multivariable generalization of Wiener-Kolmogorov
  - In 1962 Pontryagin et al published the maximal principle
  - In 1965 Isaacs published his book on differential games
Course Content

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**Problem Definition**

We will define the basic *optimal control problem*:

- **Given system dynamics**

  \[
  \dot{x} = f(x, u), \quad x \in X \subset \mathbb{R}^n, u \in U \subset \mathbb{R}^m
  \]

- **Find a control** \( u(t), \ t \in [0, T] \) that steers the system from an initial state \( x(0) = x_0 \) to a target set \( G \) and minimizes the cost

  \[
  J(u(\cdot)) = g_T(x(T)) + \int_0^T g(x(t), u(t)) \, dt
  \]

**Remark**

\( g_T \) is called the *terminal cost* and \( g \) is the *running cost*. The terminal time \( T \) can be fixed or free. The target set can be fixed or moving.
**Open Loop vs. Closed Loop**

- If we are concerned with a single specified initial state $x_0$, then we might seek the optimal control $u(t), u: \mathbb{R} \rightarrow \mathbb{R}^m$ that steers the system from the initial state to the target. This is an open loop control.

- On the other hand, we might seek the optimal control as a function of the state $u(x), u: \mathbb{R}^n \rightarrow \mathbb{R}^m$. This is a closed loop control; sometimes called a synthesis.

- The open loop control is sometimes easier to compute, and the computations are sometimes performed online – a method known as model predictive control.

- The closed loop control has the important advantage that it is robust with respect to model uncertainty, and that once the (sometimes difficult) computations are performed off-line, the control is easily implemented online.
Consider steering a unit mass, with bounded applied control force, from an arbitrary initial position and velocity to rest at the origin in minimum time. Specifically,

\[
\begin{align*}
\dot{x} &= v \\
\dot{v} &= u, \quad |u| \leq 1
\end{align*}
\]

The cost function is

\[
J = \int_0^T dt \equiv T
\]

**Remark**

This is an example of a problem with a control constraint.
**Example — a Minimum Fuel Problem**

Consider the decent of a moon lander.

\[
\begin{align*}
\dot{h} &= v \\
\dot{v} &= -g + \frac{u}{m} \\
\dot{m} &= -ku
\end{align*}
\]

The thrust \( u \) is used to steer the system to \( h = 0, v = 0 \). In addition we wish to minimize the fuel used during landing, i.e.

\[
J = \int_0^t ku \, dt
\]

Furthermore, \( u \) is constrained, \( 0 \leq u \leq c \), and the state constraint \( h \geq 0 \) must be respected.

**Remark**

*This problem has both control and state constraints.*
EXAMPLE – a LINEAR REGULATOR PROBLEM

Consider a system with linear dynamics

\[ \dot{x} = Ax + Bu \]

We seek a feedback control that steers the system from an arbitrary initial state \( x_0 \) towards the origin in such a way as to minimize the cost

\[
J = x^T(T)Qx(T) + \frac{1}{2T} \int_0^T \left\{ x^T(t)Qx(t) + u^T(t)Ru(t) \right\} dt
\]

The final time \( T \) is considered fixed.
**Example – a Robust Servo Problem**

Consider a system with dynamics

\[
\begin{align*}
\dot{x} &= Ax + B_1w + B_2u \\
z &= C_1x + D_{11}w + D_{12}u \\
y &= C_2x + D_{21}w + D_{22}u
\end{align*}
\]

where \(w\) is an external disturbance. The goal is to find an output (\(y\)) feedback synthesis such that the performance variables (process errors) \(z\) remain close to zero. Note that \(w(t)\) can be characterized in several ways, stochastic (the \(H_2\) problem)

\[
J = E \left[ \int_{-\infty}^{\infty} z^T(t) z(t) \, dt \right]
\]

or deterministic (the \(H_\infty\) problem)

\[
J = \max_{\|w\|_2=1} \int_{-\infty}^{\infty} z^T(t) z(t) \, dt
\]
DEFINITIONS

Consider the scalar function

\[ f(x), \quad x \in \mathbb{R}^n \]

which is defined and smooth on a domain \( D \subset \mathbb{R}^n \). We further assume that the region \( D \) is defined by a scalar inequality \( \psi(x) \leq 0 \), i.e.,

\[
\text{int}D = \{ x \in \mathbb{R}^n \mid \psi(x) < 0 \}
\]

\[
\partial D = \{ x \in \mathbb{R}^n \mid \psi(x) = 0 \}
\]
**DEFINITIONS**

**Definition (Local Minima & Maxima)**

An interior point \( x^* \in \text{int}D \) is a **local minimum** if there exists a neighborhood \( U \) of \( x^* \) such that

\[
f(x) \geq f(x^*) \quad \forall x \in U
\]

It is a **local maximum** if

\[
f(x) \leq f(x^*) \quad \forall x \in U
\]

Similarly, for a point \( x^* \in \partial D \), we use a neighborhood \( U \) of \( x^* \) within \( \partial D \). With this modification boundary local minima and maxima are defined as above.
Optimal Interior Points

- Necessary conditions. A point $x^* \in \text{int}D$ is an extremal point (minimum or maximum) only if
  \[ \frac{\partial f}{\partial x} (x^*) = 0 \]

- Sufficient conditions. $x^*$ is a minimum if
  \[ \frac{\partial^2 f}{\partial x^2} (x^*) > 0 \]
  a maximum if
  \[ \frac{\partial^2 f}{\partial x^2} (x^*) < 0 \]
Optimization with constraints – Necessary Conditions

We need to find extremal points of \( f(x) \), with \( x \in \partial D \). i.e., find extremal points of \( f(x) \) subject to the constraint \( \psi(x) = 0 \).

Consider a more general problem where there are \( m \) constraints, i.e., \( \psi : \mathbb{R}^n \rightarrow \mathbb{R}^m \).

Let \( \lambda \) be an \( m \)-dimensional constant vector (called Lagrange multipliers) and define the function

\[
H(x, \lambda) \triangleq f(x) + \lambda^T \psi(x)
\]

Then \( x^* \) is an extremal point only if

\[
\frac{\partial H(x^*, \lambda)}{\partial x} = 0, \quad \frac{\partial H(x^*, \lambda)}{\partial \lambda} \equiv \psi(x) = 0
\]

Note there are \( n + m \) equations in \( n + m \) unknowns \( x, \lambda \).
Consider extremal points of $f(x_1, x_2)$ subject to the single constraint $\psi(x_1, x_2) = 0$. At an extremal point $(x_1^*, x_2^*)$ we must have

$$df(x_1^*, x_2^*) = \frac{\partial f(x_1^*, x_2^*)}{\partial x_1}dx_1 + \frac{\partial f(x_1^*, x_2^*)}{\partial x_2}dx_2 = 0$$  \hspace{1cm} (1)$$

but $dx_1$ and $dx_2$ are not independent. They satisfy

$$d\psi(x_1^*, x_2^*) = \frac{\partial \psi(x_1^*, x_2^*)}{\partial x_1}dx_1 + \frac{\partial \psi(x_1^*, x_2^*)}{\partial x_2}dx_2 = 0$$  \hspace{1cm} (2)$$

From (1) and (2) it must be that

$$\frac{\partial f(x_1^*, x_2^*)}{\partial x_1} \frac{\partial \psi(x_1^*, x_2^*)}{\partial x_1} = \frac{\partial f(x_1^*, x_2^*)}{\partial x_2} \frac{\partial \psi(x_1^*, x_2^*)}{\partial x_2} \Delta = -\lambda$$

Accordingly, (1) and (2) yield

$$\frac{\partial f(x_1^*, x_2^*)}{\partial x_1} + \lambda \frac{\partial \psi(x_1^*, x_2^*)}{\partial x_1} = 0, \quad \frac{\partial f(x_1^*, x_2^*)}{\partial x_2} + \lambda \frac{\partial \psi(x_1^*, x_2^*)}{\partial x_2} = 0$$

**Remark**

*Note that these are the form of the previous slide.*
Optimization with Constraints – Sufficient Conditions

\[ df = \frac{\partial H}{\partial x} dx + \frac{1}{2} dx^T \frac{\partial^2 H}{\partial x^2} dx - \lambda^T d\psi + \text{h.o.t.} \]

but

\[ d\psi = \frac{\partial \psi}{\partial x} dx = 0 \Rightarrow dx \in \ker \frac{\partial \psi}{\partial x} \Rightarrow dx = \Psi d\alpha \]

\[ \Psi = \text{span} \ker \frac{\partial \psi (x)}{\partial x} \]

Thus, for \( x^* \) extremal

\[ df (x^*) = \frac{1}{2} d\alpha^T \Psi^T \frac{\partial^2 H (x^*)}{\partial x^2} \Psi d\alpha + \text{h.o.t.} \]

\[ \Psi^T \frac{\partial^2 H (x^*)}{\partial x^2} \Psi > 0 \Rightarrow \min \]

\[ \Psi^T \frac{\partial^2 H (x^*)}{\partial x^2} \Psi < 0 \Rightarrow \max \]
**Example**

\[ f(x_1, x_2) = x_1 \left( x_1^2 + 2x_2^2 - 1 \right), \quad \psi(x_1, x_2) = x_1^2 + x_2^2 - 1 \]

**Interior:**

\[(x_1, x_2, f) = (0, -0.707, 0) \lor (0, 0.707, 0) \lor (-0.577, 0, 0.385) \lor (0.577, 0, -0.385)\]

**Boundary:**

\[(x_1, x_2, \lambda, f) = (-0.577, -0.8165, 1.155, -0.385) \lor (-0.577, 0.8165, 1.155, -0.385) \lor (0.577, -0.8165, -1.155, 0.385) \lor (0.577, 0.8165, -1.155, 0.385) \lor (-1, 0, 1, 0) \lor (1, 0, -1, 0)\]
Optimizing a Time Trajectory

- We are interested in steering a controllable system along a trajectory that is optimal in some sense.
- Three methods are commonly used to address such problems:
  - The ‘calculus of variations’
  - The Pontryagin ‘maximal Principle’
  - The ‘principle of optimality’ and dynamic programming
- The calculus of variations was first invented to characterize the dynamical behavior of physical systems governed by a conservation law.
CALCULUS OF VARIATIONS: LAGRANGIAN SYSTEMS

A Lagrangian System is characterized as follows:

- The system is defined in terms of a vector of configuration coordinates, \( q \), associated with velocities \( \dot{q} \).
- The system has kinetic energy \( T(\dot{q}, q) = \dot{q}^T M(q) \dot{q} / 2 \), and potential energy \( V(q) \) from which we define the Lagrangian

\[
L(\dot{q}, q) = T(\dot{q}, q) - V(q)
\]

- The system moves along a trajectory \( q(t) \), between initial and final times \( t_1, t_2 \) in such a way as to minimize the integral

\[
J(q(t)) = \int_{t_1}^{t_2} L(\dot{q}(t), q(t)) \, dt
\]
**Examples of Lagrangian Systems**

\[
T = \frac{1}{2} \dot{q}^T M(q) \dot{q}
\]

\[
M(q) = \begin{pmatrix}
\ell_1^2 (m_1 + m_2) + \ell_2^2 m_2 + 2\ell_1 \ell_2 m_2 \cos \theta_2 & \ell_2 m_2 (\ell_2 + \ell_1 \cos \theta_2) \\
\ell_2 m_2 (\ell_2 + \ell_1 \cos \theta_2) & \ell_2^2 m_2
\end{pmatrix}
\]

\[
V(q) = m_1 g (g \ell_1 (m_1 + m_2) \sin \theta_1 + g \ell_2 m_2 \sin (\theta_1 + \theta_2))
\]
**NECESSARY CONDITIONS: FIXED TERMINAL TIME –1**

- A real-valued, continuously differentiable function $q(t)$ on the interval $[t_1, t_2]$ will be called *admissible*.
- Let $q^*(t)$ be an optimal admissible trajectory and $q(t, \varepsilon)$ a not necessarily optimal trajectory, with

  $$q(t, \varepsilon) = q^*(t) + \varepsilon \eta(t)$$

  where $\varepsilon > 0$ is a small parameter and $\eta(t)$ is arbitrary.
- Then

  $$J(q(t, \varepsilon)) = \int_{t_1}^{t_2} L(\dot{q}^*(t) + \varepsilon \dot{\eta}(t), q^*(t) + \varepsilon \eta(t)) \, dt$$

- An extremal of $J$ is obtained from

  $$\delta J(q(t)) = \frac{\partial J(q(t,\varepsilon))}{\partial \varepsilon} \bigg|_{\varepsilon=0} = 0$$
Fixed Terminal Time – 2

\[
\delta J (q(t)) = \int_{t_1}^{t_2} \left( \frac{\partial L (\dot{q}^*(t), q^*(t))}{\partial \dot{q}^*} \eta(t) + \frac{\partial L (\dot{q}^*(t), q^*(t))}{q^*} \right) dt
\]

We can apply the ‘integration by parts’ formula

\[
\int u dv = uv - \int v du
\]

to the first term to obtain

\[
\delta J (q(t)) = \int_{t_1}^{t_2} \left( -\frac{d}{dt} \frac{\partial L (\dot{q}^*(t), q^*(t))}{\partial q^*} + \frac{\partial L (\dot{q}^*(t), q^*(t))}{q^*} \right) \eta(t) dt
\]

\[
+ \left. \frac{\partial L (\dot{q}^*(t), q^*(t))}{\partial \dot{q}^*} \eta(t) \right|_{t_1}^{t_2}
\]
**Fixed Terminal Time – 3**

Now, set $\delta J(q(t)) = 0$ to obtain:

- the Euler-Lagrange equations

$$\frac{d}{dt} \left( \frac{\partial L(\dot{q}^*(t), q^*(t))}{\partial \dot{q}^*} \right) - \frac{\partial L(\dot{q}^*(t), q^*(t))}{q^*} = 0$$

- the transversality conditions

$$\frac{\partial L(\dot{q}^*(t_1), q^*(t_1))}{\partial \dot{q}^*} \delta q(t_1) = 0, \quad \frac{\partial L(\dot{q}^*(t_2), q^*(t_2))}{\partial \dot{q}^*} \delta q(t_2) = 0$$

**Remark**

*These results allow us to treat problems in which the initial and terminal times are fixed and individual components of $q(t_1)$ and $q(t_2)$ are fixed or free. Other cases of interest include: 1) the terminal time is free, and 2) the terminal time is related to the terminal configuration, e.g., by a relation $\varphi(q(t_2), t_2) = 0.*
**NECESSARY CONDITIONS: FREE TERMINAL TIME – 1**

Consider the case of fixed initial state and free terminal time. Let $q^*(t)$ be the optimal trajectory with optimal terminal time $t_2^*$. The perturbed trajectory terminates at time $t_2^* + \delta t_2$. Its end state is $q^*(t_2^*) + \varepsilon \eta (t_2^* + \varepsilon \tau)$. The perturbed cost is

$$ J(q^*, \delta q, \delta t) = \int_{t_1}^{t_2^* + \delta t} L(\dot{q}^*(t) + \delta \dot{q}(t), q^*(t) + \delta q(t)) \, dt $$

From this we obtain:

$$ \delta J = \int_{t_1}^{t_2^*} \left( -\frac{d}{dt} \frac{\partial L(q^*(t), q^*(t))}{\partial \dot{q}^*} + \frac{\partial L(q^*(t), q^*(t))}{q^*} \right) \delta q(t) \, dt - \frac{\partial L(q^*(t_1^*), q^*(t_1^*))}{\dot{q}^*} \delta q(t_1^*) $$

$$ + \frac{\partial L(q^*(t_2^*), q^*(t_2^*))}{\dot{q}^*} \delta q(t_2^*) + L(\dot{q}^*(t_2^* + \delta t), q^*(t_2^*)) \delta t $$

Now, we want to allow both the final time and the end point to vary. The actual end state is:

$$ \delta q_2 \triangleq \delta q(t_2^* + \delta t) = \delta q(t_2^*) + \dot{q}^*(t_2^*) \delta t $$
\[
\delta J = \int_{t_1}^{t_2} \left( -\frac{d}{dt} \frac{\partial L(\dot{q}^*(t), q^*(t))}{\partial \dot{q}^*} + \frac{\partial L(\dot{q}^*(t), q^*(t))}{q^*} \right) \delta q(t) \, dt - \frac{\partial L(\dot{q}^*(t_1^*), q^*(t_1^*))}{\partial \dot{q}^*} \delta q(t_1^*) \\
+ \frac{\partial L(\dot{q}^*(t_2^*), q^*(t_2^*))}{\partial \dot{q}^*} \delta q(t_2^*) + \left( L(\dot{q}^*(t_2^*), q^*(t_2^*)) - \frac{\partial L(\dot{q}^*(t_2^*), q^*(t_2^*))}{\partial q^*} \dot{q}^*(t_2^*) \right) \delta t
\]

Thus, we have have the Euler-Lagrange Equations, as before, but the transversality conditions become:

- the previous conditions:
  \[ \frac{\partial L(\dot{q}^*(t_1^*), q^*(t_1^*))}{\partial \dot{q}^*} \delta q(t_1^*) = 0, \quad \frac{\partial L(\dot{q}^*(t_2^*), q^*(t_2^*))}{\partial \dot{q}^*} \delta q(t_2^*) = 0 \]

- plus, additional conditions:
  \[ \left( L(\dot{q}^*(t_2^*), q^*(t_2^*)) - \frac{\partial L(\dot{q}^*(t_2^*), q^*(t_2^*))}{\partial q^*} \dot{q}^*(t_2^*) \right) \delta t = 0 \]
Ordinarily, the initial state and time are fixed so that the transversality conditions require

\[ \delta q (t_1) = 0 \]

For free terminal time \( \delta t \) is arbitrary. Thus from the transversality conditions:

- If the terminal state is fixed, \( \delta q (t_2) = 0 \) the terminal condition is:

\[ L (\dot{q}^* (t_2^*), q^* (t_2^*)) - \frac{\partial L (\dot{q}^* (t_2^*), q^* (t_2^*))}{\partial \dot{q}^*} \dot{q}^* (t_2^*) = 0 \]

- If the terminal state is free, \( \delta q_2 \) arbitrary the terminal condition is:

\[ \frac{\partial L (\dot{q}^* (t_2^*), q^* (t_2^*))}{\partial \dot{q}^*} = 0, \quad L (\dot{q}^* (t_2^*), q^* (t_2^*)) = 0 \]
CONTROL EXAMPLE

First note that in the elementary variational calculus, we could simply replace $q$ by $x$, and add the definition $\dot{x} = u$, so that the cost function becomes

$$J(x(t)) = \int_{t_1}^{t_2} L(u(t), x(t)) \, dt$$

Hence, we have a simple control problem. We consider 3 variants:

1. Free endpoint, fixed terminal time
2. Fixed endpoint, fixed terminal time
3. Fixed endpoint, free terminal time


**Example 1: Free endpoint, Fixed terminal time**

Suppose \( x \in \mathbb{R}, t_1 = 0, t_2 = 1, \) and \( L = \frac{x^2 + u^2}{2} \):

\[
J(x(t)) = \int_0^1 \frac{1}{2} (x^2 + u^2) \, dt
\]

The Euler-Lagrange equations become:

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial u} \right) - \frac{\partial L}{\partial x} = 0 \Rightarrow \dot{u} - x = 0
\]

The transversality conditions are:

\[
x(0) = x_0, \quad \frac{\partial L(x(1), u(1))}{\partial u} = 0
\]
EXAMPLE 1, CONT’D

Thus, we have the equations

\[
\begin{bmatrix}
\dot{x} \\
\dot{u}
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
x \\
u
\end{bmatrix}
\]

\[x(0) = x_0, \quad \frac{\partial L(x(1), u(1))}{\partial u} = 0 \Rightarrow u(1) = 0\]

Consequently,

\[
\begin{bmatrix}
x(t) \\
u(t)
\end{bmatrix} = e^{t
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
}
\begin{bmatrix}
a \\
b
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
x(t) \\
u(t)
\end{bmatrix} =
\begin{bmatrix}
a \cosh(t) + b \sinh(t) \\
b \cosh(t) + a \sinh(t)
\end{bmatrix}
\]

\[a = x_0, \quad b = \frac{x_0 - e^2x_0}{1 + e^2}\]
Example 1: Fixed endpoint, Fixed terminal time

Suppose we consider a fixed end point, say \( x(t_2) = 0 \) with the terminal time still fixed at \( t_2 = 1 \). Then the Euler-Lagrange equations remain the same, but the boundary conditions change to:

\[
x(0) = x_0, \quad x(1) = 0
\]

Thus, we compute

\[
a = x_0, \quad b = -\frac{x_0 + e^2 x_0}{-1 + e^2}
\]
Example 1: Fixed endpoint, Free terminal time

Suppose we consider a fixed end point, say $x(t_2) = x_f$ with the terminal time $t_2$ free. Then the Euler-Lagrange equations remain the same, but the boundary conditions change to:

$$x(0) = x_0, \quad x(t_2) = x_f, \quad (L - L_u u)|_{t_2} = 0$$

From this we compute

$$a = x_0, \quad a^2 = b^2, \quad a \cosh(t_2) + b \sinh(t_2) = x_f$$
**Example 1, Cont’d**

The figures below show the optimal control and state trajectories from the initial state $x_0 = 1, x_f = 0$, with $t_2 = 1$ for the fixed time case. For the free time case $x_f = 0.01$.

**Remark**

*In the free time case, with $x_f = 0.01$, the final time is $t_2 = 4.60517$. With $x_f = 0$ the final time is $t_2 = \infty$. Note that the case of free terminal state and free terminal time (not shown) is trivial with $t_2 = 0$.***
**Variational Calculus with Differential Constraints**

Systems with ‘nonintegrable’ differential, i.e., nonholonomic, constraints have been treated by variational methods. As before, we seek extremals of the functional:

\[ J(q(t)) = \int_{t_1}^{t_2} L(\dot{q}(t), q(t), t) \, dt \]

But now subject to the constraints:

\[ \varphi(\dot{q}(t), q(t), t) = 0 \]

where \( \varphi : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m \).
Differential Constraints, Cont’d

The constraints are enforce for all time, so introduce the $m$ Lagrange multipliers $\lambda(t)$ and consider the modified functional

$$J(q(t)) = \int_{t_1}^{t_2} L(\dot{q}(t), q(t), t) + \lambda^T(t) \varphi(\dot{q}(t), q(t), t) \, dt$$

We allow variations in both $q$ and $\lambda$ and $t_2$, Taking the variation and integrating by parts yields

$$\delta J = \int_{t_1}^{t_2} \left( \left\{ -\frac{d}{dt} \left[ L_q + \lambda^T \varphi_q \right] + \left[ L_q + \lambda^T \varphi_q \right] \right\} \delta q(t) + \varphi^T \delta \lambda \right) \, dt$$

$$+ \left[ L_\dot{q} + \lambda^T \varphi_\dot{q} \right] t_2 \delta q_2 + \left( \left[ L + \lambda^T \varphi_q \right] - \left[ L_\dot{q} + \lambda^T \varphi_\dot{q} \right] \dot{q} \right)_{t_2} \delta t_2$$
**NECESSARY CONDITIONS WITH CONSTRAINTS**

Again we have the Euler-Lagrange equations, now in the form:

\[
\frac{d}{dt} [L\dot{q} + \lambda^T \varphi \dot{q}] - [Lq + \lambda^T \varphi q] = 0
\]

The differential constraints:

\[
\varphi (\dot{q}, q, t) = 0
\]

and the boundary conditions

\[
[L\dot{q} + \lambda^T \varphi \dot{q}]_{t_2} \delta q_2 = 0
\]

With free terminal time, we also have

\[
([L + \lambda^T \varphi q] - [L\dot{q} + \lambda^T \varphi \dot{q}] \dot{q})_{t_2} = 0
\]
**Example**

Once again consider the problem

\[
\dot{x} = u, \quad J(x(t)) = \int_0^{t_2} \frac{1}{2} \left( x^2 + u^2 \right) dt
\]

But now, add the constraint

\[
\varphi(\dot{x}, x) = \dot{x}^2 - 1 \equiv u^2 - 1 = 0 \rightarrow u = \pm 1
\]

The Euler-Lagrange equations now become

\[
\frac{d}{dt} \left[ Lu + \lambda^T \varphi_u \right] - \left[ Lx + \lambda^T \varphi_x \right] = 0 \Rightarrow \dot{\lambda} = \frac{x}{2u}
\]

**Remark**

- Note that \( u = \pm 1 \rightarrow \dot{u} = 0 \) *almost everywhere*,
- Also, \( \int_0^t u^2 dt = t \)
We now consider three cases:

- free end point, fixed terminal time
- fixed end point, fixed terminal time
- fixed end point, free terminal time
EXAMPLE: FREE END POINT, FIXED TERMINAL TIME

Steer from $x(0) = x_0$.

- system $\dot{x} = u$
- Euler equation $\dot{\lambda} = \frac{x}{2u}$
- constraint $u = \pm 1$
- initial condition $x(0) = x_0$
- terminal condition $[L_u + \lambda^T \varphi_u]_{t_2} = 0 \Rightarrow \lambda(1) = -\frac{1}{2}$

The necessary conditions are satisfied with

$$(k = \pm 1) \land \left(-\frac{3}{4} \leq \lambda_0 \leq -\frac{1}{2}\right) \land \left(T = \frac{1}{2} - \frac{\sqrt{1 - 2\lambda_0}}{\sqrt{2}}\right)$$
Example: Free End Point, Fixed terminal Time—2

Only the case $\lambda_0 = -\frac{1}{2}, T = \frac{1}{2}$ is optimal.

\[
\lambda_0 = -\frac{1}{2}
\]

\[
\lambda_0 = -\frac{3}{4}
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Introduction

Optimal Control Problem

Optimization Basics

Variational Calculus

Systems with Constraints

Example: Fixed End Point, Fixed terminal Time

Steer from \( x(0) = x_0 \) to \( x(1) = x_1 \).

- system \( \dot{x} = u \)
- Euler equation \( \dot{\lambda} = \frac{x}{2u} \)
- constraint \( u = \pm 1 \)
- boundary conditions \( x(0) = x_0, x(1) = x_1 \)

Assume single switch at time \( T \)

\[
x_0 + kT - k(1 - T) = x_1 \land (k = \pm 1) \land 0 \leq T \leq 1
\Rightarrow -1 + x_1 \leq x_0 \leq 1 + x_1 \land (k = \pm 1) \land T = \frac{k-x_0+x_1}{2k}
\]
EXAMPLE: FREE TERMINAL TIME, FIXED END POINT

Steer from \( x(0) = x_0 \) to \( x(t_2) = 0 \)

- system \( \dot{x} = u \)
- Euler equation \( \dot{\lambda} = \frac{x}{2u} \)
- constraint \( u = \pm 1 \)
- initial condition \( x(0) = x_0, x(t_2) = 0 \)
- terminal time condition \( (L + \lambda^T \varphi_x) - (Lu - \lambda^T \varphi_u) = 0 \Rightarrow \lambda(t_2) = \frac{1}{4k} + \frac{1}{2} \)

\[
(x_0 \geq 0, t_2 \geq x_0) \land (k = \pm 1) \land \left( T = \frac{t_2 - kx_0}{2} \right) \land \left( \lambda_0 = \frac{(t_2 - 2T)^2 - 1}{4} \right)
\]

\[
(x_0 \leq 0, t_2 \geq -x_0) \land (k = \pm 1) \land \left( T = \frac{t_2 - kx_0}{2} \right) \land \left( \lambda_0 = \frac{(t_2 - 2T)^2 - 1}{4} \right)
\]