Manifolds, Vector Fields & Flows

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Outline

- From Flat to Curved State Space – Why?
- Manifolds
- Maximum rank condition
- Regular manifolds
- Tangent space & tangent bundle
- Differential map
- Vector fields & flows
- Lie bracket
From Flat to Curved State Space

- Global models may require it:
  - Mechanical systems spatial rotation
  - Electric power systems DAE description
- Correct local approximations at least require acknowledging it
- Even if not required it may have conceptual benefits
- Computation often involves flat local approximations – but this may not be necessary (e.g. quaternions vs. Euler angles)
Representations of Surfaces

- Explicit \( y = g(x) \)
- Implicit \( f(x, y) = 0 \)
- Parametric \( x = h_1(s), y = h_2(s), s \in U \subset R \)

\[
\begin{align*}
y &= \pm \sqrt{1-x^2} \\
x^2 + y^2 - 1 &= 0 \\
x &= \cos s, \ y = \sin s, \ s \in [0, 2\pi)
\end{align*}
\]
Examples: Parametrically defined manifolds

\[ f = \begin{bmatrix} \cos t \sin u \\ \sin t \sin u \\ \cos u \end{bmatrix} \quad t \in [0, 2\pi), u \in [0, \pi) \]

\[ f = \begin{bmatrix} \cos t(3 + \cos u) \\ \sin t(3 + \cos u) \\ \sin u \end{bmatrix} \quad t \in [0, 2\pi), u \in [0, 2\pi) \]
Definition - Manifold

An \( m \)-dimensional manifold is a set \( M \) together with a countable collection of subsets \( U_i \subset M \) and one-to-one mappings onto open subsets of \( \mathbb{R}^m \), \( \varphi_i : U_i \to V_i \), with the following properties:

- the pair \( (U_i, \varphi_i) \) is called a coordinate chart
- the coordinate chartes cover \( M \),
- on the overlap of any pair of charts the composite map is a smooth function

\[ f = \varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j) \]

- if \( p \in U_i \) and \( q \in U_j \) are distinct points of \( M \), then there are neighborhoods \( W \) of \( \varphi_i(p) \in V_i \) and \( U \) of \( \varphi_j(q) \in V_j \) such that

\[ \varphi_i^{-1}(W) \cap \varphi_j^{-1}(U) = \emptyset \]
Definition: Manifold

\[ f = \varphi_j \circ \varphi_i^{-1} \]
Example: Planet Earth
Example: Circle

The unit circle \( S_1 = \{(x,y) \mid x^2+y^2=1\} \) can be viewed as a one-dimensional manifold with two coordinate charts. Define the charts \( U_1=S_1-\{(-1,0)\} \) and \( U_2=S_1-\{(1,0)\} \). Now we define the coordinate maps by projection as shown in the figure.
Submanifold & Immersion

**Definition:** Let $F: \mathbb{R}^m \rightarrow \mathbb{R}^n$ be a smooth map. The *rank* of $F$ at $x_0 \in \mathbb{R}^m$ is the rank of the Jacobian $D_x F$ at $x_0$. $F$ is of *maximal rank* on $S \subset \mathbb{R}^m$ if the rank of $F$ is maximal for each $x_0 \in S$.

**Definition:** A *(smooth) submanifold* of $\mathbb{R}^n$ is a subset $M \subset \mathbb{R}^n$, together with a smooth one-to-one map $\phi: \Pi \subset \mathbb{R}^m \rightarrow M$ which satisfies the maximal rank condition everywhere, where the *parameter space* is $\Pi$ and $M = \phi(\Pi)$ is the image of $\phi$. If the maximal rank condition holds but the mapping is not one-to-one, then $M$ is an *immersion*.

Note that a submanifold of the space $\mathbb{R}^n$ is a parametrically defined surface.
Pathologies
Regular Submanifold

**Definition:** A *regular submanifold* $N$ of $\mathbb{R}^n$ is a submanifold parameterized by a smooth mapping $\phi$ such that $\phi$ maps homeomorphically onto its image, i.e., for each $x \in N$ there exists neighborhoods $U$ of $x$ in $\mathbb{R}^n$ such that $\phi^{-1}[U \cap N]$ is a connected open subset of the parameter space.
Implicitly Defined Regular Manifolds

**Proposition:** Consider a smooth mapping $F: \mathbb{R}^m \to \mathbb{R}^n$, $n \leq m$. If $F$ is of maximal rank on the set $S = \{x: F(x) = 0\}$, then $S$ is a regular, $(m-n)$-dimensional submanifold of $\mathbb{R}^m$.

**Example**

$$f(x, y) = (x^2 + y^2 - 1)y$$

$$Df = \begin{bmatrix} 2xy & -1 + x^2 + 3y^2 \end{bmatrix}$$

singular points: $(\pm 1, 0), (0, \pm 1/\sqrt{3})$
**The Tangent Space**

**Definition:** Let $p: \mathbb{R} \to M$ be a $C^k, k \geq 1$ map so that $p(t)$ is a curve in $M$. The *tangent vector* $v$ to the curve $p(t)$ at the point $p_0 = p(t_0)$ is defined by

$$v = \dot{p}(t_0) = \lim_{t \to t_0} \left\{ \frac{p(t) - p(t_0)}{t - t_0} \right\}$$

The set of tangent vectors to all curves in $M$ passing through $p_0$ is the *tangent space* to $M$ at $p_0$, denoted $TM_{p_0}$. 
Tangent Space / Implicit Manifold

If M is an implicit submanifold of dimension m in $\mathbb{R}^{m+k}$, i.e., $F: \mathbb{R}^{m+k} \rightarrow \mathbb{R}^k$, $M = \{x \in \mathbb{R}^{m+k} \mid F(x) = 0\}$ and $DF$ satisfies the maximum rank condition on $M$, then $T_M x$ is the $\ker D_x F(x)$ (translated, of course to the point $x$). That is $T_M p$ is the tangent hyperplane to $M$ at $p$. 
Tangent Vectors

**Definition:** The *components* of the tangent vector \( v \) to the curve \( p(t) \) in \( M \) in local coordinates \((U, \varphi)\) are the \( m \) numbers \( v_1, \ldots, v_m \) where \( v_i = \frac{d \varphi(p(t))}{dt} \).

Consider the map \( F : M \to \mathbb{R} \). Let \( y = f(x), x \in \varphi(U) \subset \mathbb{R}^m \) denote the realization of \( F \) in the local coordinates \((U, \varphi)\). Again, \( p(t) \) denotes a curve in \( M \) with \( x(t) \) its image in \( \mathbb{R}^m \). Then the rate of change of \( F \) at a point \( p \) on this curve is

\[
\frac{df}{dt} = v_1 \frac{\partial f}{\partial x_1} + \cdots + v_m \frac{\partial f}{\partial x_m}
\]
Tangent Vectors as Derivations

The tangent vector $v = [v_1 \cdots v_m]$ is uniquely determined by the action of the directional derivative operator (called a derivation)

$$v = v_1 \frac{\partial}{\partial x_1} + \cdots v_m \frac{\partial}{\partial x_m}$$
Natural Basis

\[ v = \begin{bmatrix} 0 \\ \\ \\ 1 \\ \\ \\ 0 \end{bmatrix} \]

\[ v = \begin{bmatrix} 1_i \end{bmatrix} \]

Definition: The set of partial derivative operators constitute a basis for the tangent space $TM_p$ for all points $p \in U \subset M$ which is called the natural basis.

The natural coordinate system on $TM_p$ induced by $(U, \varphi)$ has basis vectors that are tangent vectors to the coordinate lines on $M$ passing through $p$. 
**Tangent Bundle**

**Definition:** The union of all the tangent spaces to M is called the tangent bundle and is denoted TM,

\[ TM = \bigcup_{p \in M} TM_p \]

**Remark:** The tangent bundle is a manifold with \( \dim TM = 2 \dim M \). A point in \( TM \) is a pair \((x,v)\) with \( x \in M, \ v \in TM_x \). If \((x_1,..,x_m)\) are local coordinates on M and \((v_1,..,v_m)\) components of the tangent vector in the natural coordinate system on \( TM_x \), then natural local coordinates on TM are

\[
(x_1,\ldots,x_m,v_1,\ldots,v_m) = (x_1,\ldots,x_m,\dot{x}_1,\ldots,\dot{x}_m)
\]

Recall the natural ‘unit vectors’ on \( TM_x \) are

\[
v_1 = \partial/\partial x_1 ,\ldots, \ v_m = \partial/\partial x_m
\]
Summary

- Regular Manifold
  - Parametrically defined
  - Implicitly defined
- Tangent Space, Tangent Vector, Tangent Bundle

\[ M = \{ x \in \mathbb{R}^3 \mid f_1(x_1, x_2, x_3) = 0 \} \quad \text{rank} \left[ \frac{\partial f_1}{\partial x_1} \quad \frac{\partial f_1}{\partial x_2} \right] = 1 \text{ on } M \]
Mechanical System State Space

A *mechanical system* is a collection of mass particles which interact through physical constraints or forces. A *configuration* is a specification of the position for each of its constituent particles. The *configuration space* is a set $M$ of elements such that any configuration of the system corresponds to a unique point in the set $M$ and each point in $M$ corresponds to a unique configuration of the system. The configuration space of a mechanical system is a differentiable manifold called the *configuration manifold*. Any system of local coordinates $q$ on the configuration manifold are called *generalized coordinates*. The *generalized velocities* $\dot{q}$ are elements of the tangent spaces to $M$, $TM_q$. The *state space* is the tangent bundle $TM$ which has local coordinates $(q, \dot{q})$. 
Example: Pendulum
Given the map $F : M \rightarrow N$, the \textit{differential map} is the induced mapping

$$F_* : TM_p \rightarrow TN_{F(p)}$$

that takes tangent vectors into tangent vectors.
Differential Map ~ local coordinates

In local coordinates, the chain rule yields

\[ \frac{d\bar{\phi}}{dt} = \frac{\partial F}{\partial x} \frac{d\phi}{dt} \Rightarrow \bar{v} = \frac{\partial F}{\partial x} v \]

• The map \( F_* \) is also denoted \( dF \)
• The Jacobian is the representation of the differential map in local coordinates
Vector Fields

**Definition:** A *vector field* $\nu$ on $M$ is a map which assigns to each point $p \in M$, a tangent vector $\nu(p) \in TM_p$. It is a $C^k$-vector field if for each $p \in M$ there exist local coordinates $(U, \varphi)$ such that each component $\nu_i(x)$, $i=1,..,m$ is a $C^k$ function for each $x \in \varphi(U)$.

**Definition:** An *integral curve* of a vector field $\nu$ on $M$ is a parameterized curve $p = \phi(t)$, $t \in (t_1, t_2) \subset \mathbb{R}$ whose tangent vector at any point coincides with $\nu$ at that point.
Integral Curves

In local coordinates \((U, \varphi)\), the image of an integral curve \(x(t) = \varphi \circ \phi(t)\) satisfies the ode

\[
\frac{dx}{dt} = v(x)
\]
Flow

**Definition:** Let \( v \) be a smooth vector field on \( M \) and denote the parameterized maximal integral curve through \( p \in M \) by \( \Psi(t,p) \) and \( \Psi(0,p) = p \). \( \Psi(t,p) \) is called the *flow generated by \( v \).*

**Properties of flows:**

- satisfies ode \( \frac{d}{dt} \Psi(t,p) = v(\Psi(t,p)), \quad \Psi(0,p) = p \)

- semigroup property \( \Psi(t_2, \Psi(t_1,p)) = \Psi(t_1 + t_2, p) \)
Exponential Map

We will adopt the notation

\[ e^{tv} p := \Psi(t, p) \]

The motivation for this is that the flow satisfies the three basic properties ordinarily associated with exponentiation – from properties of \( \Psi(t, p) \).

\[ e^{0v} p = p \quad \text{boundary condition} \]

\[ \frac{d}{dt} e^{tv} p = v(e^{tv} p) \quad \text{differential equation} \]

\[ e^{(t_1 + t_2)v} p = e^{t_1v} e^{t_2v} p \quad \text{semi-group property} \]
Series Expansion Along Trajectory

Suppose \( x(t) \) satisfies \( \dot{x} = v(x), \ x(0) = x_0 \). Let \( f : \mathbb{R}^m \to \mathbb{R}^p \).

\[
f(x(t)) = f(x_0) + \left[ \frac{d}{dt} f(x(t)) \right]_t t + \frac{1}{2} \left[ \frac{d^2}{dt^2} f(x(t)) \right]_{t=0} t^2 + \cdots
\]

\[
\frac{d}{dt} f(x(t)) = \frac{\partial f}{\partial x} v(x) = v_1(x) \frac{\partial f}{\partial x_1} + \cdots + v_m(x) \frac{\partial f}{\partial x_m} = v(f)(x)
\]

\[
\frac{d^2}{dt^2} f(x(t)) = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} v(x) \right) v(x) = \mathbf{v}^2 (f)(x)
\]

\[
f(x(t)) = f(x_0) + \mathbf{v}(f)(x_0) t + \frac{1}{2} \mathbf{v}^2 (f)(x_0) t^2 + \cdots
\]
Series Representation of Exp Map

For $f$ a scalar or vector, we can derive the Taylor expansion of $f(x(t))$ about $t=0$

$$f\left(e^{t\nu}x\right) = \sum_{k=0}^{\infty} \frac{t^k}{k!} v^k (f)(x)$$

Choose $f(x)=x$, to obtain

$$e^{t\nu}x = \sum_{k=0}^{\infty} \frac{t^k}{k!} v^k (x)(x)$$
Example: scalar linear fields

\[ \nu = 1 \Rightarrow \frac{dx}{dt} = 1 \Rightarrow \Psi(t, x) = x + t \]

\[ \Psi(t, x) = e^{t \frac{\partial}{\partial x}} x = \left(1 + t \frac{\partial}{\partial x} + \frac{1}{2} t^2 \frac{\partial^2}{\partial x^2} + \cdots \right) x = x + t \]

\[ \nu = x \Rightarrow \frac{dx}{dt} = x \Rightarrow \Psi(t, x) = e^t x \]

\[ \Psi(t, x) = e^{tx \frac{\partial}{\partial x}} x = \sum_{k=0}^{\infty} \frac{t^k}{k!} v^k(x)(x) = \sum_{k=0}^{\infty} \frac{t^k}{k!} x = e^t x \]
Example: general linear field

\[ v(x) = Ax = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} x \Rightarrow v = a_1 x \frac{\partial}{\partial x_1} + \cdots + a_n x \frac{\partial}{\partial x_n} \]

\[ v(x) = a_1 x \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + a_n x \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = Ax \]

\[ v^2 (x) = v(Ax) = A v(x) = A^2 x \]

\[ v^k (x) = v(A^{k-1}x) = A^{k-1} v(x) = A^k x \]

\[ e^{t x} = \sum_{k=0}^{\infty} \frac{t^k}{k!} v^k (x)(x) = \left( \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k x \right) = e^{Ax} \]
Example: Affine Field

\[ v(x) = Ax + b = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} x + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \Rightarrow v = (a_1 x + b_1) \frac{\partial}{\partial x_1} + \cdots + (a_n x + b_n) \frac{\partial}{\partial x_n} \]

\[ v(x) = (a_1 x + b_1) \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + (a_n x + b_n) \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = Ax + b \]

\[ v^k(x) = v(A^{k-1} x + A^{k-2} b) = A^{k-1} v(x) = A^k x + A^{k-1} b \]

\[ e^{Tv} x = \sum_{k=0}^{\infty} \frac{t^k}{k!} v^k(x)(x) = \left( \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k x + \sum_{k=0}^{\infty} \frac{t^k}{k!} A^{k-1} b \right) = e^{At} x + e^{At} A^{-1} b \]
Examples, Cont’d

\[ v(x) = -x^3 \iff \dot{x} = -x^3 \]

Exact solution:

\[ x(t) = \frac{x_0}{\sqrt{1 + 2x_0^2t}} = \left(1 - x_0^2t + \frac{12}{8} x_0^4 t^2 - \frac{20}{8} x_0^6 t^3 + \frac{35}{8} x_0^8 t^4 - \frac{63}{8} x_0^{10} t^5 + \cdots \right)x_0 \]

via exponential map:

\[ x(t) = e^{x_0^3 \partial / \partial x} x_0 = \left(1 - x_0^2t + \frac{12}{8} x_0^4 t^2 - \frac{20}{8} x_0^6 t^3 + \frac{35}{8} x_0^8 t^4 - \frac{63}{8} x_0^{10} t^5 + \cdots \right)x_0 \]
**Lie Derivative**

Definition: Let $v(x)$ denote a vector field on $M$ and $F(x)$ a mapping from $M$ to $\mathbb{R}^n$, both in local coordinates. Then the *Lie derivative of order* $0, \ldots, k$ is

$$L^0_v(F) = F, \quad L^k_v(F) = \frac{\partial L^{k-1}_v}{\partial x} v$$

With this notation we can write

$$v^k(F)(x) = L^k_v(F)(x)$$
Example: Exponential Map of a Nonlinear Field

\[ v(x) = \begin{bmatrix} v_1(x) \\ v_2(x) \\ \vdots \\ v_n(x) \end{bmatrix} \Rightarrow \mathbf{v} = v_1(x) \frac{\partial}{\partial x_1} + \cdots + v_n(x) \frac{\partial}{\partial x_n} \]

\[ \mathbf{v}(x) = L_v(x) = v(x) \]

\[ \mathbf{v}^2(x) = \mathbf{v}(v(x)) = L_v^2(x) \]

\[ \mathbf{v}^k(x) = \mathbf{v}(A^{k-1}x) = A^{k-1} \mathbf{v}(x) = L_v^k(x) \]

\[ e^{t \mathbf{v}} x = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathbf{v}^k(x) = \left( \sum_{k=0}^{\infty} \frac{t^k}{k!} L_v^k(x) \right) \]
**Lie Bracket**

**Definition:** If \( v, w \) are vector fields on \( M \), then their *Lie bracket* \([v, w]\) is the unique vector field defined in local coordinates by the formula

\[
[v, w] = \frac{\partial w}{\partial x} v - \frac{\partial v}{\partial x} w
\]

**Property:**

\[
\frac{dw(\Psi(t, x))}{dt} \bigg|_{t=0} = [v, w]_x
\]

The rate of change of \( w \) along the flow of \( v \)
Let us consider the Lie bracket as a commutator of flows. Beginning at point \( x \) in \( M \) follow the flow generated by \( v \) for an infinitesimal time which we take as \( \sqrt{\varepsilon} \) for convenience. This takes us to point

\[
y = \exp(\sqrt{\varepsilon} \cdot v) x
\]

Then follow \( w \) for the same length of time, then \(-v\), then \(-w\). This brings us to a point \( \psi \) given by

\[
\psi(\varepsilon, x) = e^{-\sqrt{\varepsilon}w} e^{-\sqrt{\varepsilon}v} e^{\sqrt{\varepsilon}w} e^{\sqrt{\varepsilon}v} x
\]
Lie Bracket Interpretation
Continued

\[ \frac{d}{d\varepsilon} \Psi(0^+, x) = \left[ v, w \right]_x \]
Summary

- Definition of regular manifold
  - Implicitly defined & parametrically defined
- Local coordinates
- Tangent space, vector field, integral curve
- Differential map, exponential map
- Lie derivative
- Lie bracket