Outline

- Ordinary Differential Equations
  - Existence & uniqueness
  - Continuous dependence on parameters
  - Invariant sets, nonwandering sets, limit sets

- Lyapunov Stability
  - Autonomous systems
  - Basic stability theorems
  - Stable, unstable & center manifolds
  - Control Lyapunov function
Basics of Nonlinear ODE’s
A solution on a time interval \( t \in [t_0, t_1] \) is a function \( x(t) : [t_0, t_1] \rightarrow \mathbb{R}^n \) that satisfies the ode.
Vector Fields and Flow

- We can visualize an individual solution as a graph $x(t): t \rightarrow \mathbb{R}^n$.
- For autonomous systems it is convenient to think of $f(x)$ as a vector field on $\mathbb{R}^n$ - $f(x)$ assigns a vector to each point in $\mathbb{R}^n$. As $t$ varies, a solution $x(t)$ traces a path through $\mathbb{R}^n$ tangent to the field $f(x)$.
- These curves are often called trajectories or orbits.
- The collection of all trajectories in $\mathbb{R}^n$ is called the flow of the vector field $f(x)$. 
Auto at Constant Speed

\[
\frac{d}{dt} \begin{bmatrix}
\omega \\
\beta
\end{bmatrix} = f(\omega, \beta, \bar{V}, \delta)
\]

Notice the three equilibria.
Van der Pol

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
x_2 \\
-0.8(1-x_1^2)x_2 - x_1
\end{bmatrix}
\]
Damped Pendulum

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
x_2 \\
-x_2 / 2 - \sin x_1
\end{bmatrix}
\]
Lipschitz Condition

The existence and uniqueness of solutions depend on properties of the function $f$. In many applications $f(x,t)$ has continuous derivatives in $x$. We relax this - we require that $f$ is **Lipschitz** in $x$.

**Def:** $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz on an open subset $D \subset \mathbb{R}^n$ if each point $x_0 \in D$ has a neighborhood $U_0$ such that

$$
\|f(x) - f(x_0)\| \leq L\|x - x_0\|
$$

for some constant $L$ and all $x \in U_0$

Note: $C^0$ (continuous) functions need not be Lipschitz, $C^1$ functions always are.
The Lipschitz Condition

- A Lipschitz continuous function is limited in how fast it can change,
- A line joining any two points on the graph of this function will never have a slope steeper than its Lipschitz constant $L$,
- The mean value theorem can be used to prove that any differentiable function with bounded derivative is Lipschitz continuous, with the Lipschitz constant being the largest magnitude of the derivative.
Examples: Lipschitz

\[ f(x) = x^2, \quad x \in [-3, 7] \]
\[ L = 14 \]

\[ f(x) = \sqrt{x^2 + 1}, \quad x \in \mathbb{R} \]
\[ L = 1 \]

\[ f(x) = |x - 1|, \quad x \in [-5, 5] \]

\[ f(x) = \sqrt{x}, \quad x \in [0, 4] \]

Fails
Local Existence & Uniqueness

**Proposition** (Local Existence and Uniqueness) Let $f(x,t)$ be piece-wise continuous in $t$ and satisfy the Lipschitz condition

$$\|f(x,t) - f(y,t)\| \leq L\|x - y\|$$

for all $x, y \in B_r = \{x \in \mathbb{R}^n \|x - x_0\| < r\}$ and all $t \in [t_0, t_1]$. Then there exists $\delta > 0$ such that the differential equation with initial condition

$$\dot{x} = f(x,t), \quad x(t_0) = x_0 \in B_r$$

has a unique solution over $[t_0, t_0 + \delta]$. 
The Flow of a Vector Field

\[ \dot{x} = f(x), \quad x(t_0) = x_0 \Rightarrow x(x_0, t) \] this notation indicates 'the solution of the ode that passes through \( x_0 \) at \( t = 0 \)

More generally, let \( \Psi(x, t) \) denote the solution that passes through \( x \) at \( t = 0 \). The function \( \Psi : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n \) satisfies

\[
\frac{\partial \Psi(x, t)}{\partial t} = f(\Psi(x, t)), \quad \Psi(x, 0) = x
\]

\( \Psi \) is called the flow or flow function of the vector field \( f \)
Example: Flow of a Linear Vector Field

\[ \dot{x} = Ax \Rightarrow \frac{\partial \Psi(x,t)}{\partial t} = A \Psi(x,t) \Rightarrow \Psi(x,t) = e^{At} x \]

Example:

\[
\begin{bmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & -1
\end{bmatrix}
\]

\[
x \in \mathbb{R}^3, \quad A = \begin{bmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & -1
\end{bmatrix}, \quad \Psi(x,t) = \begin{bmatrix}
x_1 \cos(t) + x_2 \sin(t) \\
x_2 \cos(t) - x_1 \sin(t) \\
e^{-t} x_3
\end{bmatrix}
\]
Invariant Set

A set of points $S \subset R^n$ is invariant with respect to the vector field $f$ if trajectories beginning in $S$ remain in $S$ both forward and backward in time.

Examples of invariant sets:
- any entire trajectory (equilibrium points, limit cycles)
- collections of entire trajectories
Example: Invariant Set

\[
\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
\]

- each of the three trajectories shown are invariant sets
- the $x_1$-$x_2$ plane is an invariant set
Limit Points & Sets

A point $q \in \mathbb{R}^n$ is called an $\omega$-limit point of the trajectory $\Psi(t, p)$ if there exists a sequence of time values $t_k \to +\infty$ such that

$$\lim_{t_k \to \infty} \Psi(t_k, p) = q$$

$q$ is said to be an $\alpha$-limit point of $\Psi(t, p)$ if there exists a sequence of time values $t_k \to -\infty$ such that

$$\lim_{t_k \to -\infty} \Psi(t_k, p) = q$$

The set of all $\omega$-limit points of the trajectory through $p$ is the $\omega$-limit set, and the set of all $\alpha$-limit points is the $\alpha$-limit set.

For an example see invariant set
Introduction to Lyapunov Stability Analysis
Lyapunov Stability

\( \dot{x} = f(x), \quad f(0) = 0, \)

\( f : D \to \mathbb{R}^n \) (locally Lipschitz)

The origin is

- a **stable** equilibrium point if for each \( \varepsilon > 0 \), there is a \( \delta(\varepsilon) > 0 \) such that
  \[
  \|x(0)\| < \delta \implies \|x(t)\| < \varepsilon \quad \forall t > 0
  \]
- **unstable** if it is not stable, and
- **asymptotically stable** if \( \delta \) can be chosen such that
  \[
  \|x(0)\| < \delta \implies \lim_{t \to \infty} x(t) \to 0
  \]
Two Simple Results

The origin is asymptotically stable only if it is isolated.

The origin of a linear system

$$\dot{x} = Ax$$

is stable if and only if

$$\|e^{At}\| \leq N < \infty \forall t > 0$$

It is asymptotically stable if and only if, in addition

$$\|e^{At}\| \to 0, \; t \to \infty$$
Example: Non-isolated Equilibria

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
x_2 \\ -x_2 \mid x_1 - x_2
\end{bmatrix}
\]

All points on the \( x_1 \) axis are equilibrium points.
Positive Definite Functions

A function $V : R^n \rightarrow R$ is said to be

- positive definite if $V(0) = 0$ and $V(x) > 0$, $x \neq 0$
- positive semi-definite if $V(0) = 0$ and $V(x) \geq 0$, $x \neq 0$
- negative (semi-) definite if $-V(x)$ is positive (semi-) definite
- radially unbounded if $V(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$

For a quadratic form: $V(x) = x^T Q x$, $Q = Q^T$ the following are equivalent

- $V(x)$ is positive definite
- the eigenvalues of $Q$ are positive
- the principal minors of $Q$ are positive
Lyapunov Stability Theorem

\( V(x) \) is called a Lyapunov function relative to the flow of \( \dot{x} = f(x) \) if it is positive definite and nonincreasing with respect to the flow:

\[
V(0) = 0, \quad V(x) > 0 \text{ for } x \neq 0
\]

\[
\dot{V} = \frac{\partial V(x)}{\partial x} f(x) \leq 0
\]

**Theorem:** If there exists a Lyapunov function on some neighborhood \( D \) of the origin, then the origin is stable. If \( \dot{V} \) is negative definite on \( D \) then it is asymptotically stable.
Example: Linear System

\[ \dot{x} = Ax \]
\[ V(x) = x^T Q x, \quad Q^T = Q > 0 \]
\[ \dot{V} = x^T Q \dot{x} + \dot{x}^T Q x = x^T (QA + A^T Q) x = -x^T Px \]

where \( QA + A^T Q = -P \) \quad \text{Lyapunov Equation}

\[ P > 0 \Rightarrow A \text{ is stable (Hurwitz): } \|e^{At}\| \text{ bounded } \forall t > 0 \]

So we can specify \( Q \), compute \( P \) and test \( P \).
Or, specify \( P \) and solve Lyapunov equation for \( Q \) and test \( Q \).
Example: Rotating Rigid Body

$x, y, z$ body axes; $\omega_x, \omega_y, \omega_z$ angular velocities in body coord's;

$\text{diag}(I_x, I_y, I_z) I_x \geq I_y \geq I_z > 0$ inertia matrix

$$
\begin{align*}
\ddot{\omega}_x &= -\left( \frac{I_z - I_y}{I_x} \right) \omega_z \omega_y = a \omega_z \omega_y \\
\ddot{\omega}_y &= -\left( \frac{I_x - I_z}{I_y} \right) \omega_x \omega_z = -b \omega_x \omega_z \quad \text{note } a, b, c > 0 \\
\ddot{\omega}_z &= -\left( \frac{I_y - I_x}{I_z} \right) \omega_y \omega_x = c \omega_y \omega_x
\end{align*}
$$
A state \((\bar{\omega}_x, \bar{\omega}_y, \bar{\omega}_z)\) is an equilibrium point if any two of the angular velocity components are zero, i.e., the \(\omega_x, \omega_y, \omega_z\) axes are all equilibrium points.

Consider a point \((\bar{\omega}_x, 0, 0)\). Shift \(\omega_x \rightarrow \omega_x + \bar{\omega}_x\).

\[
\begin{align*}
\dot{\omega}_x &= a \omega_z \omega_y \\
\dot{\omega}_y &= -b (\omega_x + \bar{\omega}_x) \omega_z \\
\dot{\omega}_z &= c (\omega_x + \bar{\omega}_x) \omega_y
\end{align*}
\]
Rigid Body, Cont’d

Energy does not work for $\ddot{\omega}_x \neq 0$. Obvious?

So, how do we find Lyapunov function?

We want

$$V(0,0,0) = 0,$$

$$V(\omega_x, \omega_y, \omega_z) > 0 \text{ if } (\omega_x, \omega_y, \omega_z) \neq (0,0,0) \text{ and } (\omega_x, \omega_y, \omega_z) \in D \text{ (some neighborhood of the origin)}$$

$$\dot{V} \leq 0$$

Lets look at all functions that satisfy $\dot{V} = 0$, i.e., that satisfy the pde:

$$\frac{\partial V}{\partial \omega_x} \omega_x \omega_y + \frac{\partial V}{\partial \omega_y} \left( -b(\omega_x + \ddot{\omega}_x) \omega_z \right) + \frac{\partial V}{\partial \omega_z} c(\omega_x + \ddot{\omega}_x) \omega_y = 0$$

All solutions take the form:

$$V(\omega_x, \omega_y, \omega_z) = cA + bB + (cA - bB)^2 = \frac{1}{2} \left( \frac{8b^2c^2 \ddot{\omega}_x^2}{a^2} \omega_x^2 + c\omega_y^2 + b\omega_z^2 \right) + \text{h.o.t.}$$
Rigid Body, Cont’d

Clearly,

\[ V(0) = 0, \ V > 0 \] on a neighborhood \( D \) of 0

\[ \dot{V} = 0 \]

\( \Rightarrow \) spin about \( x \)-axis is stability

- This is one approach to finding candidate Lyapunov functions
- The first order PDE usually has many solutions
- The method is connected to traditional ‘first integral’ methods to the study of stability in mechanics
- Same method can be used to prove stability for spin about \( z \)-axis, but spin about \( y \)-axis is unstable – why?
**LaSalle Invariance Theorem**

**Theorem:** Suppose $V : R^n \to R$ is $C^1$ and let $\Omega_c$ denote a component of the region

$$\left\{ x \in R^n \mid V(x) < c \right\}$$

Suppose $\Omega_c$ is bounded and within $\Omega_c$, $\dot{V}(x) \leq 0$.

Let $E$ be the set of points within $\Omega_c$ where $\dot{V}(x) = 0$.

Let $M$ be the largest invariant set within $E$.

$\Rightarrow$ every sol'n beginning in $\Omega_c$ tends to $M$.

as $t \to \infty$. 
Example: LaSalle’s Theorem

\[
\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -m^{-1}kx_1 - m^{-1}cx_2 \end{bmatrix},
\]

\[
V(x_1, x_2) = \frac{1}{2} mx_2^2 + \frac{1}{2} kx_1^2
\]

\[
\dot{V}(x_1, x_2) = -cx_2^2
\]
Lagrangian Systems

\[ \frac{d}{dt} \frac{\partial L(\dot{x}, x)}{\partial \dot{x}} - \frac{\partial L(\dot{x}, x)}{\partial x} = Q^T \]

\( x \in \mathbb{R}^n \) generalized coordinates

\( \dot{x} = dx / dt \) generalized velocities

\( L : \mathbb{R}^{2n} \to \mathbb{R} \) is the Lagrangian, \( L(\dot{x}, x) = T(\dot{x}, x) - U(x) \)

kinetic energy: \( T(\dot{x}, x) = \frac{1}{2} \dot{x}^T M(x) \dot{x} \)

total energy: \( V(x, \dot{x}) = T(\dot{x}, x) + U(x) \)
Lagrange-Poincare Systems

\[ \dot{x} = p, \quad \frac{d}{dt} \frac{\partial L(p,x)}{\partial p} - \frac{\partial L(p,x)}{\partial x} = Q^T \]

\[ L(p,x) = T(p,x) - U(x), \quad T(p,x) = \frac{1}{2} p^T M(x) p \]

\[ \frac{\partial L(p,x)}{\partial p} = p^T M(x), \quad \frac{d}{dt} \frac{\partial L(p,x)}{\partial p} = \dot{p}^T M(x) + p^T \frac{\partial p^T M(x)}{\partial x} \]

\[ \dot{p}^T M(x) + p^T \frac{\partial p^T M(x)}{\partial x} - p^T \frac{\partial M(x)}{\partial x} p + \frac{\partial U(x)}{\partial x} = Q^T \]
Example

\[ \dot{x}_1 = x_2, \quad T = \frac{x_2^2}{2}, \quad U = \frac{x_1^2}{1 + x_1^2} \]

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix}
= \begin{bmatrix}
0 & x_2 \\
-2 \frac{x_1}{(1 + x_1^2)^2} & -cx_2
\end{bmatrix}
\]

\[ V(x) = \frac{x_2^2}{2} + \frac{x_1^2}{1 + x_1^2}, \quad \dot{V}(x) = -cx_2 \leq 0 \text{ for } c > 0 \]

Notice that

- the level sets are unbounded for \( V(x) = \text{constant} \geq 1 \)
- \( V(x) \) is not radially unbounded
Example, Cont’d
First Integrals

**Definition:** A *first integral* of the differential equation

\[ \dot{x} = f(x, t) \]

is a scalar function \( \varphi(x, t) \) that is constant along trajectories, i.e.,

\[ \dot{\varphi}(x, t) = \frac{\partial \varphi(x, t)}{\partial x} f(x, t) + \frac{\partial \varphi(x, t)}{\partial t} = 0 \]

**Observation:** For simplicity, consider the autonomous case \( \dot{x} = f(x) \). Suppose \( \varphi_1(x) \) is a first integral and \( \varphi_2(x), \ldots, \varphi_n(x) \) are arbitrary independent functions on a neighborhood of the point \( x_0 \), i.e.,

\[ \det \frac{\partial}{\partial x} \left[ \begin{array}{c} \varphi_1(x) \\ \vdots \\ \varphi_n(x) \end{array} \right] \Bigg|_{x = x_0} \neq 0 \]

Then we can define coordinate transformation \( x \to z \), via

\[ z = \varphi(x) \Rightarrow \dot{z} = \left[ \frac{\partial \varphi(x)}{\partial x} f(x) \right]_{x = \varphi^{-1}(z)} \Rightarrow \dot{z}_1 = 0 \Rightarrow z_1 \equiv \text{constant} \]

The problem has been reduced to solving \( n - 1 \) differential equations.
Noether’s Theorem

If the Lagrangian is invariant under a smooth 1 parameter
cchange of coordinates, $h_s : M \rightarrow M$, $s \in R$, then the Lagrangian
system has a first integral

$$\Phi (p,q) = \left. \frac{dL}{dp} \frac{dh_s(q)}{ds} \right|_{s=0}$$

$$T(p,q) = \frac{1}{2} \begin{bmatrix} v & \omega \end{bmatrix} \begin{bmatrix} M+m & m\ell \cos \theta & \ell \cos \theta & \ell \sin \theta \\ ml \cos \theta & m\ell^2 & ml \cos \theta & ml \sin \theta \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}, \quad U(q) = -mg \ell \cos \theta$$

$$h_s (X, \theta) = \begin{bmatrix} X + s \\ 0 \end{bmatrix}, \quad \Phi (p,q) = (M+m)v$$

Momentum in X
direction
Chetaev’s Method

Consider the system of equations
\[ \dot{x} = f(x,t), \quad f(0,t) = 0 \]
We wish to study the stability of the equilibrium point \( x = 0 \).
Obviously, if \( \varphi(x,t) \) is a first integral and it is also a positive definite function, then \( V(x,t) = \varphi(x,t) \) establishes stability. But suppose \( \varphi(x,t) \) is not positive definite?

Suppose the system has \( k \) first integrals \( \varphi_1(x,t), \ldots, \varphi_k(x,t) \) such that \( \varphi_i(0,t) = 0 \).
Chetaev suggested the construction of Lyapunov functions of the form:
\[ V(x,t) = \sum_{i=1}^{k} \alpha_i \varphi_i(x,t) + \sum_{i=1}^{k} \beta_i \varphi_i^2(x,t) \]
Chetaev Instability Theorem

Let $D$ be a neighborhood of the origin. Suppose there is a function $V(x) : D \rightarrow \mathbb{R}$ and a set $D_1 \subset D$ such that

1) $V(x)$ is $C^1$ on $D$,
2) the origin belongs to the boundary of $D_1$, $\partial D_1$,
3) $V(x) > 0$, $\dot{V}(x) > 0$ on $D_1$,
4) on the boundary of $D_1$ inside $D$, i.e., on $\partial D_1 \cap D$, $V(x) = 0$

Then the origin is unstable.
Consider the rigid body with spin about the y-axis (intermediate inertia), \( \vec{\omega} = (0, \bar{\omega}_y, 0)^T \)

\[
\dot{\omega}_x = a \omega_z (\omega_y + \bar{\omega}_y)
\]

Shifted equations:

\[
\begin{align*}
\dot{\omega}_y &= -b \omega_x \omega_z \\
\dot{\omega}_z &= c (\omega_y + \bar{\omega}_y) \omega_x
\end{align*}
\]

Attempts to prove stability fails. So, try to prove instability.

Consider \( V(\omega_x, \omega_y, \omega_z) = \omega_x \omega_z \)

Let \( B_r = \left\{ (\omega_x, \omega_y, \omega_z) \mid \omega_x^2 + \omega_y^2 + \omega_z^2 < r^2 \right\} \) and \( D_1 = \left\{ (\omega_x, \omega_y, \omega_z) \in B_r \mid \omega_x > 0, \omega_z > 0 \right\} \)

so that \( V > 0 \) on \( D_1 \) and \( V = 0 \) on \( \partial D_1 \)

\[
\dot{V} = a \omega_z^2 (\omega_y + \bar{\omega}_y) + c (\omega_y + \bar{\omega}_y) \omega_x^2 = (\omega_y + \bar{\omega}_y)(a \omega_z^2 + c \omega_x^2)
\]

We can take \( r^2 < \bar{\omega}_y^2 \Rightarrow \omega_y + \bar{\omega}_y > 0 \forall (\omega_x, \omega_y, \omega_z) \in B_r \)

in which case \( \dot{V} > 0 \) on \( D_1 \Rightarrow \text{instability} \)
Stability of Linear Systems - Summary

Consider the linear system
\[ \dot{x} = Ax \]

Choose \( V(x) = x^T P x \)

\[ \Rightarrow \dot{V}(x) = x^T (A^T P + PA) x := -x^T Q x \]

a) if there exists a positive definite pair of matrices \( P, Q \) that satisfy

\[ A^T P + PA = -Q \quad \text{(Lyapunov equation)} \]

the origin is asymptotically stable.

b) if \( P \) has at least one negative eigenvalue and \( Q > 0 \), the origin is unstable.

c) if the origin is asymptotically stable then for any \( Q > 0 \), there is a unique solution, \( P > 0 \), of the Lyapunov equation.
Second Order Systems

Consider the system

\[ M\ddot{x} + C\dot{x} + Kx = 0, \quad M^T = M > 0, \quad C^T = C > 0, \quad K^T = K > 0 \]

\[ E(\dot{x}, x) = \frac{1}{2} \dot{x}^T M\dot{x} + \frac{1}{2} x^T Kx \]

\[ \frac{d}{dt} E(\dot{x}, x) = \dot{x}^T M\dot{x} + x^T K\dot{x} = -\dot{x}^T [C\dot{x} + Kx] + x^T K\dot{x} \]

\[ \frac{d}{dt} E(\dot{x}, x) = -\dot{x}^T C\dot{x} \]

Some interesting generalizations:
1) \( C \geq 0 \), 2) \( C^T \neq C \), 3) \( K^T \neq K \)

The anti-symmetric terms correspond to ‘gyroscope’ forces – they are conservative.

The anti-symmetric terms correspond to ‘circulatory’ forces (transfer conductances in power systems) – they are non-conservative.
Example

- Assume uniform damping
- Assume $e=0$
- Designate Gen 1 as swing bus
- Eliminate internal bus 4

\[
\dot{\theta}_1 + \gamma \dot{\theta}_1 = \Delta P_1 - b_{13} \sin(\theta_1) - b_{12} \sin(\theta_1 - \theta_2)
\]
\[
\dot{\theta}_2 + \gamma \dot{\theta}_2 = \Delta P_2 + b_{12} \sin(\theta_1 - \theta_2) - b_{23} \sin(\theta_2)
\]
\[
\theta_1 = \delta_2 - \delta_1, \quad \theta_2 = \delta_3 - \delta_1, \quad \Delta P_1 = P_2 - P_1, \quad \Delta P_2 = P_3 - P_1
\]
Example Cont’d

- This is a Lagrangian system with
  \[ U(\theta_1, \theta_2) = -\Delta P_1 \theta_1 - \Delta P_2 \theta_2 - b_{11} \cos(\theta_1) - b_{12} \cos(\theta_1 - \theta_2) - b_{23} \cos(\theta_2) \]
  \[ T(\omega_1, \omega_2) = \frac{1}{2}(\omega_1^2 + \omega_2^2), \quad Q = [-\gamma \omega_1 \quad -\gamma \omega_2] \]

- To study stability choose total energy as Lyapunov function
  \[ V = T(\omega_1, \omega_2) + U(\theta_1, \theta_2) \]
  \[ \dot{V} = -\gamma \omega_1^2 - \gamma \omega_2^2 \leq 0 \]
  Note: \( T(0, 0) = 0 \) and \( T(\omega_1, \omega_2) > 0 \quad \forall (\omega_1, \omega_2) \neq 0 \quad \Rightarrow \)
  Equilibria corresponding to \( U(\theta_1, \theta_2) \) a local minimum are stable.

Example, Cont’d

Since $-\pi \leq \theta_1 < \pi$ and $-\pi \leq \theta_2 < \pi$ we should consider $U(\theta_1, \theta_2)$ as a function on a torus $U : \mathcal{T} \to \mathbb{R}$

$P_1 = 0$, $P_2 = 0$, $b_{12} = 1$, $b_{13} = 1$, $b_{23} = 1$
Example, Cont’d

\[ P_1 = .25, \ P_2 = 0, \ b_{12} = 1, \ b_{13} = 1, \ b_{23} = 1 \]
$P_1 = \pi / 15, \ P_2 = 0, \ b_{12} = 1, \ b_{13} = 1, \ b_{23} = 1$
Example Cont’d

$$P_1 = \pi / 5, \ P_2 = -1, \ b_{12} = 1, \ b_{13} = .5, \ b_{23} = 1$$
Control Lyapunov Function

Consider the controlled system

\[ \dot{x} = f(x, u), \quad x \in D \subset R^n \text{ containing } x=0, u \in R^m \]

Find \( u(x) \) such that all trajectories beginning in \( D \) converge to \( x = 0 \).

**Definition**: A control Lyapunov function (CLF) is a function \( V: D \to R \) with \( V(0) = 0, \ V(x) > 0, \ x \neq 0, \) such that

\[ \forall x \neq 0 \in D, \exists u(x) \quad \dot{V}(x) = \frac{\partial V}{\partial x} f(x, u(x)) < 0 \]

**Theorem**: (Artsteins Theorem) A differentiable CLF exists iff there exists a 'regular' feedback control \( u(x) \).
Example

Consider the nonlinear mass-spring-damper system:

\[ v = \dot{q}, \quad m(1 + q^2)\dot{v} + bv + k_0q + k_1q^3 = u \]

\[
\frac{d}{dt} \begin{bmatrix} q \\ v \end{bmatrix} = \begin{bmatrix} \frac{v}{1 + q^2} \\ \frac{1}{m(1 + q^2)}(-k_0q - k_1q^3 - bv + u) \end{bmatrix}
\]

suppose the target state is \( v = 0, q = 0 \),

Define \( r^2 = v^2 + \alpha q^2, \alpha > 0 \). A CLF candidate is \( V = \frac{1}{2}r^2 \). This is positive definite wrt the target state

\[ \dot{V} = r\dot{r} = (\dot{v} + \alpha q)v = v \left\{ \frac{u - bv - k_0q - k_1q^3}{m(1 + q^2)} + \alpha q \right\} = -\kappa v^2, \quad \kappa > 0 \]

\[ \left\{ \frac{u - bv - k_0q - k_1q^3}{m(1 + q^2)} + \alpha q \right\} = -\kappa v \leq 0 \quad \Rightarrow \quad u = bv + k_0q + k_1q^3 - m(1 + q^2)(\alpha q + \kappa v) \]

\[ \Rightarrow \text{closed loop} \quad \frac{d}{dt} \begin{bmatrix} q \\ v \end{bmatrix} = \begin{bmatrix} v \\ -\kappa v - \alpha q \end{bmatrix} \]