Robust Control 8
Robust Stability

Harry G. Kwatny
Department of Mechanical Engineering & Mechanics
Drexel University

5/13/2003
Outline

• Modeling Uncertainty
• Robust Stability/Robust Performance
• Robust Stability of the $M-\Delta$ Structure
• Uncertainty using Coprime Factors
• The Structured Singular Value
Modeling Uncertainty

Let $\Delta(s)$ be a stable transfer matrix with $\|\Delta\|_\infty \leq \gamma$

$\Delta$ may be unstructured, i.e., full
$\Delta$ may be structured, e.g., block diagonal or some elements may be real.

Additive uncertainty: $G_p = G + \Delta$

Multiplicative uncertainty:

$G_p = G(I + \Delta)$ input

$G_p = (I + \Delta)G$ output
Example

A spacecraft attitude control model that includes the rigid body dynamics plus one flexible mode is given by the transfer function

\[ G_p(s) = \frac{s^2 + 2s + 2}{s^2(s^2 + s + 1)} \]

A nominal model that includes only the rigid body dynamics is

\[ G(s) = \frac{2}{s^2} \]

Additive uncertainty:

\[ G_p(s) = G(s) + \Delta \Rightarrow \Delta = \frac{s^2 + 2s + 2}{s^2(s^2 + s + 1)} - \frac{2}{s^2} = \frac{-1}{s^2 + s + 1} \]

Multiplicative uncertainty:

\[ G_p(s) = (1 + \Delta)G(s) \Rightarrow \Delta = \frac{G_p - G}{G} = \frac{-s^2}{2(s^2 + s + 1)} \]
M-Δ Structure

\[ M = N_{11} \]

\[ N \overset{\Delta}{=} F = P_{11} + P_{12} \left( I - P_{22} K \right)^{-1} P_{21} \]
Obtaining $P, N, M$

To generate $P$, note
inputs: $u_\Delta, w, u$
outputs: $y_\Delta, z, y$

$$P = \begin{bmatrix} 0 & 0 & I \\ WG & W & WG \\ -G & -I & -G \end{bmatrix} \Rightarrow N = \begin{bmatrix} -KG(I + KG)^{-1} & -K(I + GK)^{-1} \\ WG(I + KG)^{-1} & W(I + GK)^{-1} \end{bmatrix}$$
Singular Values

Recall a singular value and the corresponding singular vectors of a rectangular matrix $A$ are, respectively, the scalar $\sigma$ and the two vectors $u, v$ that satisfy

$$Av = \sigma u$$

$$A^Tu = \sigma v$$

The singular value decomposition of $A$ is

$$A = U\Sigma V^T$$
Robust Stability/Robust Performance

The perturbed system closed loop transfer function $w \to z$ is

$$F_\Delta = N_{22} + N_{21} \Delta (I - N_{11} \Delta)^{-1} N_{12}$$

Nominal Stability (NS) $\iff N$ is stable
Nominal Performance (NP) $\iff \|N_{22}\|_\infty < 1; \ & \text{NS}$
Robust Stability (RS) $\iff F_\Delta$ is stable $\forall \Delta, \|\Delta\|_\infty \leq 1; \ & \text{NS}$
Robust Performance (RP) $\iff \|F_\Delta\|_\infty < 1, \forall \Delta, \|\Delta\|_\infty \leq 1; \ & \text{NS}$
Small Gain Theorem

Recall spectral radius: \( \rho(L(j\omega)) = \max_i |\lambda_i(L(j\omega))| \)

Theorem (Spectral radius stability theorem). Consider a system with stable open loop transfer function \( L(s) \). Then the closed loop is stable if \( \rho(L(j\omega)) < 1, \text{ } \forall \omega \).

Theorem (Small Gain Theorem). Consider a system with stable open loop transfer function \( L(s) \). Then the closed loop is stable if \( \|L(j\omega)\| < 1, \text{ } \forall \omega \), where \( \|L\| \) denotes any norm satisfying \( \|AB\| \leq \|A\| \cdot \|B\| \).
Determinant Stability Condition

**Theorem:** Assume $M(s),\Delta(s)$ are stable and $\Delta$ belongs to a convex set of perturbations, such that if $\Delta'$ is a member then so is $c\Delta'$ for any real scalar $c$ with $|c| \leq 1$. Then the $M - \Delta$ system is stable for all admissible perturbations if and only if

The Nyquist plot of $\det(I - M\Delta)$ does not encircle the origin for all $\Delta$

$\iff \det(I - M\Delta(j\omega)) \neq 0, \forall \omega, \forall \Delta$

$\iff \lambda_i(M\Delta(j\omega)) \neq 1, \forall \omega, \forall \Delta, i = 1, \ldots, \dim(M\Delta)$
Spectral Radius Stability Condition

**Theorem:** Assume \( M(s), \Delta(s) \) are stable and \( \Delta \) belongs to a set of perturbations, such that if \( \Delta' \) is a member then so is \( c\Delta' \) for any complex scalar \( c \) with \( |c| \leq 1 \). Then the \( M - \Delta \) system is stable for all admissible perturbations if and only if

\[
\rho(M\Delta(j\omega)) < 1, \quad \forall \omega, \forall \Delta
\]

\[\uparrow\]

\[
\max_{\Delta} \rho(M\Delta(j\omega)) < 1, \quad \forall \omega
\]
Robust Stability of $M-\Delta$ Structure

Theorem (Robust stability for unstructured perturbations). Assume that the nominal system $M(s)$ is stable and that the perturbations $\Delta(s)$ are stable. Then the $M-\Delta$ system is stable for all perturbations satisfying $\|\Delta\|_\infty \leq 1$ if and only if

$$\bar{\sigma}(M(j\omega)) < 1, \quad \forall \omega \iff \|M\|_\infty < 1$$

Remarks on Proof: If $\Delta$ satisfies $\bar{\sigma}(\Delta) \leq 1$, then

$$\max_\Delta \rho(M\Delta) = \max_\Delta \bar{\sigma}(M\Delta) = \max_\Delta \bar{\sigma}(M) \bar{\sigma}(\Delta) = \bar{\sigma}(M).$$

This gives necessity. The small gain theorem with $L = M\Delta$ gives sufficiency.
Coprime Factorization

A transfer matrix can be written in (left or right) coprime factored form

\[ G = D_{\ell}^{-1}N_{\ell} = N_{r}D_{r}^{-1} \]

where

- \( N, D \) are stable, i.e., \( N \) contains all of the RHP-zeros of \( G \) and \( D \) contains all of the RHP-poles of \( G \) as RHP-zeros.
- \( N, D \) are coprime, i.e., they have no common RHP-zeros which result in cancelation. Formally, they satisfy the Bezout identity: there exist stable \( U, V \) that satisfy

\[ N_{l}U + D_{l}V = I \quad \text{or} \quad UN_{r} + VD_{r} = I \]

- a coprime factorization is called normalized if

\[ N_{l}N_{l}^{*} + D_{l}D_{l}^{*} = I \quad \text{or} \quad N_{r}^{*}N_{r} + D_{r}^{*}D_{r} = I \]
Example

\[ G(s) = \frac{(s-1)(s+2)}{(s-3)(s+4)} \]

An obvious factorization is

\[ N(s) = \frac{s-1}{s+4}, \quad D(s) = \frac{s-3}{s+2} \]

Another one is

\[ N(s) = \frac{(s-1)(s+2)}{s^2 + a_1 s + a_0}, \quad D(s) = \frac{(s-3)(s+4)}{s^2 + a_1 s + a_0}, \quad a_1, a_0 > 0 \]
Computing Coprime Factors

\[ G(s) \Leftrightarrow \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \text{ a minimal realization} \]

A normalized left coprime factorization is given by

\[
\begin{bmatrix} N_l(s) & D_l(s) \end{bmatrix} \Leftrightarrow \begin{bmatrix} A + HC & B + HD & H \\ R^{-1/2}C & R^{-1/2}D & R^{-1/2} \end{bmatrix}
\]

where

\[ H = -\left( BD^T + ZC^T \right) R^{-1}, \quad R = I + DD^T \]

and \( Z \) is the unique positive definite solution to the ARE

\[
\left( A - BS^{-1}D^TC \right) Z + Z \left( A - BS^{-1}D^TC \right)^T - ZC^T R^{-1} CZ + BS^{-1}B^T = 0
\]

with

\[ S = I + D^T D \]
Uncertainty Using Coprime Factors

Recall that a transfer matrix can be written in coprime factored form

\[ G = D^{-1}_\ell N_\ell = N_r D^{-1}_r \]

Suppose we allow separate perturbations in the numerator and denominator so that the actual plant is

\[ G_p = (D_\ell + \Delta_D)^{-1} (N_\ell + \Delta_N), \quad \|[\Delta_N \quad \Delta_D]\|_\infty \leq \varepsilon \]

This is equivalent to a system in \( M - \Delta \) structure with

\[ \Delta = [\Delta_N \quad \Delta_D], \quad M = -\begin{bmatrix} K \\ I \end{bmatrix} (I + G K)^{-1} D_\ell^{-1} \]

\[ \Rightarrow \text{RS} \quad \forall \|[\Delta_N \quad \Delta_D]\|_\infty \leq \varepsilon \iff \|M\|_\infty < 1/\varepsilon \]
Uncertainty Using Coprime Factors, Continued

\[
\begin{bmatrix}
\Delta_N \\
\Delta_D
\end{bmatrix}
\]

\[
\begin{bmatrix}
K \\
I
\end{bmatrix}
(I + GK)^{-1} D_{\ell}^{-1}
\]
M-$\Delta$ Structure for Coprime Uncertainty

This is equivalent to a system in $M - \Delta$ structure with

$$\Delta = \begin{bmatrix} \Delta_N & \Delta_D \end{bmatrix}, \quad M = -\begin{bmatrix} K \\ I \end{bmatrix} (I + GK)^{-1} D_{\ell}^{-1}$$

$$y = Gu - D_{\ell}^{-1} \Delta_D y + D_{\ell}^{-1} \Delta_N u, u = -Ky \Rightarrow y = -GKy - D_{\ell}^{-1} \Delta_D y - D_{\ell}^{-1} \Delta_N Ky$$

$$y = -\left(I + GK\right)^{-1} D_{\ell}^{-1} \begin{bmatrix} \Delta_N \\ \Delta_D \end{bmatrix} \begin{bmatrix} K \\ I \end{bmatrix} y$$

$$y = -\left(I + GK\right)^{-1} D_{\ell}^{-1} u_\Delta, u_\Delta = \begin{bmatrix} \Delta_N \\ \Delta_D \end{bmatrix} y_\Delta, y_\Delta = \begin{bmatrix} K \\ I \end{bmatrix} y$$
Controller Design with Coprime Uncertainty

Consider the family of perturbed plants

\[ G_p = \left\{ \left( D_\ell + \Delta_D \right)^{-1} \left( N_\ell + \Delta_N \right) \right\} \left\| \begin{bmatrix} \Delta_N & \Delta_D \end{bmatrix} \right\|_\infty < \varepsilon \}

with some stability margin \( \varepsilon > 0 \).

The system is robustly stabilized by the controller \( u = Ky \) if and only if

\[ \gamma \triangleq \left\| \begin{bmatrix} K \\ I \end{bmatrix} \left( I - GK \right)^{-1} D_\ell^{-1} \right\|_\infty \leq \frac{1}{\varepsilon} \]

Robust Stabilizer Design Problem: Find the lowest achievable \( \gamma \) and the corresponding maximum stability margin \( \varepsilon \) and the corresponding controller \( K \).
Solution to RSDP

\[ \gamma_{\text{min}} = \varepsilon_{\text{max}}^{-1} = (1 + \rho(XZ))^{1/2} \]

where \( X, Z \) are the unique positive definite sol'ns of the ARE's

\[
\left( A - B S^{-1} D^T C \right) Z + Z \left( A - B S^{-1} D^T C \right)^T - Z C^T R^{-1} C Z + B S^{-1} B^T = 0
\]

\[
\left( A - B S^{-1} D^T C \right)^T X + X \left( A - B S^{-1} D^T C \right) - X B S^{-1} B^T X + C^T R^{-1} C = 0
\]

and

\[ G(s) \Leftrightarrow \text{minimal realization} \ (A, B, C, D) \]

\[ R = I + D D^T, \quad S = I + D^T D \]
Solution to RSDP, Cont’d

a solution that guarantees

\[
\left\| \begin{bmatrix} K \\ I \end{bmatrix} (I - GK)^{-1} D_\ell^{-1} \right\|_\infty \leq \gamma \text{ for some } \gamma > \gamma_{\text{min}}
\]

is given by

\[
K = \begin{bmatrix}
A + BF + \gamma^2 \left( L^T \right)^{-1} ZC^T (C + DF) & \gamma^2 \left( L^T \right)^{-1} ZC^T \\
B^T X & -D^T
\end{bmatrix}
\]

\[
F = -S^{-1} \left( D^T C + B^T X \right)
\]

\[
L = \left( 1 - \gamma^2 \right) I + XZ
\]
Example: Combustion Stabilization

\[ G(s) = \frac{s + z_f}{s + p_f} \frac{(s^2 + 2\rho_z\omega_z s + \omega_z^2)}{(s^2 - 2\rho_1\omega_1 s + \omega_1^2)(s^2 + 2\rho_2\omega_2 s + \omega_2^2)} \]

\[ z_f = 1500, \quad p_f = 1000, \quad \rho_z = -0.045, \quad \omega_z = 4500 \]
\[ \rho_1 = 0.5, \quad \omega_1 = 1000, \quad \rho_2 = 0.3, \quad \omega_2 = 3500 \]

Control Design

• We will consider 3 designs
  - Classical approach – lead plus notch
  - Robust stabilization
  - Robust performance (loop shaped stabilization)

• In each case look at
  - Root locus
  - Sensitivity function
  - Error response to command step
  - Gain and phase margins
Classical Lead + Notch

\[ G_c(s) = 25 \frac{(s + 0.05)(s^2 + s + 0.7)}{(s + 5)(s^2 + s + 36)} \]
Classical Sensitivity Function

Dode Diagram
Classical Error Response to Step
Classical Margins

GMFrequency: [0.9109 4.0482 Inf]
GainMargin: [0.7418 1.7252 Inf]
PMFrequency: [0.6182 3.6938 5.2243 7.3372]
PhaseMargin: [-55.1366 25.4364 179.2394 31.6964]
Classical Nyquist
Robust Stabilization (Coprime Uncertainty)

```matlab
>> cd C:\Projects\Examples
>> combustion
gammin =
    5.1270
>> zpk(K)
Zero/pole/gain:
    -32.5492 (s+0.9894) (s-0.3953) (s^2 + 2.101s + 12.25)
-----------------------------------------------
(s+1.511) (s^2 + 9.959s + 32.55) (s^2 + 1.209s + 14.59)
```
Robust Stabilization Root Locus
Robust Stabilization Sensitivity

Bode Diagram

Magnitude (dB)

Phase (deg)

Frequency (rad/sec)
Robust Stabilization Margins

GMFrequency: [0 1.0251 2.6936 7.0934]
GainMargin: [1.8545 0.6793 1.7478 19.1030]
PMFrequency: [0.5216 1.6291]
PhaseMargin: [-29.1102 19.5459]
Robust Stabilization Nyquist
Robust Stabilization Error Response to Step
Loop Shaping for Performance: Closed Loop Transfer Functions

Assume the closed loop is stable. Then:
1. disturbance rejection $\Leftrightarrow \bar{\sigma}(S) \ll 1$
2. noise attenuation $\Leftrightarrow \bar{\sigma}(T) \ll 1$
3. reference tracking $\Leftrightarrow \bar{\sigma}(T) \approx \sigma(T) \approx 1$
4. control energy $\Leftrightarrow \bar{\sigma}(KS) \ll 1$
5. robust stability with additive uncertainty $\Leftrightarrow \bar{\sigma}(KS) \ll 1$
6. robust stability with mult output uncertainty $\Leftrightarrow \bar{\sigma}(T) \ll 1$
Loop Shaping for Performance: Open Loop Transfer Function

Assume the closed loop is stable. Then:
1. disturbance rejection $\Leftrightarrow \sigma(L) >> 1$
2. noise attenuation $\Leftrightarrow \bar{\sigma}(L) << 1$
3. reference tracking $\Leftrightarrow \sigma(L) >> 1$
4. control energy $\Leftrightarrow \bar{\sigma}(K) << 1$
5. robust stability with additive uncertainty $\Leftrightarrow \bar{\sigma}(K) << 1$, if $\bar{\sigma}(L) << 1$
6. robust stability with mult output uncertainty $\Leftrightarrow \bar{\sigma}(L) << 1$
Effect of Weighting Function

\[ W_1(s) = \frac{s+1}{s} \]

Shaped plant

Original plant
Robust Performance (Shaped Coprime Uncertainty)

```matlab
>> combustion2
gammin =
       6.7266
>> zpk(K)
Zero/pole/gain:
   -54.29 (s^2 - 0.102s + 0.1931) (s^2 + 2.101s + 12.25)
----------------------------------------
   (s+1.517) (s^2 + 11.74s + 42.92) (s^2 + 1.122s + 15.35)
```
Robust Stabilization Root Locus
Robust Stabilization Sensitivity
Robust Stabilization Margins

GMFrequency: [0.4591 1.1951 2.7718 7.5896]
GainMargin: [5.0727 0.7377 1.5541 14.4468]
PMFrequency: [0.2363 0.7681 1.7705]
PhaseMargin: [89.2292 -32.9393 14.9418]
Robust Stabilization Step Response
Robust Stabilization Nyquist

Nyquist Diagram

[Graph showing Nyquist plot with the axes labeled as Imaginary Axis and Real Axis.]
Combustion Example Summary

• Classical design does stabilize, but margins and performance are poor.
• Robust stabilization dramatically improves margins, but performance is still poor.
• Robust performance dramatically improves performance, but margins are reduced.
• Can additional loop shaping improve margins?
• Can $\gamma$-iteration improve design?
Structured Singular Value ($\mu$)

The goal is to find a generalization of $\bar{\sigma}, \rho$ that allows generalization of the above results to structured $\Delta$. Here is one approach.

Find the smallest $\Delta$ (infinity-norm) in a structured class $\mathcal{D}$ that makes $\det(I - M\Delta) = 0$; then $\mu(M) := 1/\bar{\sigma}(\Delta)$

That is

$$\mu(M)^{-1} \triangleq \min_{\Delta \in \mathcal{D}} \left\{ \bar{\sigma}(\Delta) \mid \det(I - M\Delta) = 0 \right\}$$

Notice that for an unstructured class, the smallest $\Delta$ with $\det(I - M\Delta) = 0$ has $\bar{\sigma}(\Delta) = 1/\bar{\sigma}(M) \Rightarrow \mu(M) = \bar{\sigma}(M)$
Robust Stability: Structured Uncertainty

Theorem (Robust stability for structured perturbations).
Assume that $M(s), \Delta(s)$ are stable. Then the $M - \Delta$ system is stable for all admissible perturbations satisfying $\|\Delta\|_\infty \leq 1$ if and only if $\mu(M(j\omega)) < 1, \quad \forall \omega$.

Recall: $RS \iff \det(I - M\Delta(j\omega)) \neq 0, \forall \omega, \forall \Delta, \|\Delta\|_\infty \leq 1$

If $\mu(M) < 1$ at all frequencies, the required perturbation $\Delta$ to make $\det(I - M\Delta) = 0$ is larger than 1, so the system is stable.

If $\mu(M) = 1$ at some frequency, there exists a perturbation with $\sigma(\Delta) = 1$ such that $\det(I - M\Delta) = 0$ at this frequency and the system is unstable.