Notes on Linear Robust Control

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1 Introduction to the Robust Control Problem
2 State Space Models

2.1 Solutions of Linear Systems

\[ \dot{x} = A(t)x + B(t)u \]

\[ x(t; x_0, u) = \Phi(t, t_0) x_0 + \int_{0}^{t} \Phi(t, s) B(s) u(s) ds \]

\[ \frac{d}{dt} \Phi(t, \tau) = A(t) \Phi(t, \tau), \quad \Phi(t, t) = I \]

2.2 The Matrix Exponential

Cayley-Hamilton Theorem: Every square matrix satisfies its own characteristic equation, i.e. if

\[ \phi(\lambda) = |\lambda I - A| = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_0 \]

Then

\[ \phi(A) = A^n + a_{n-1}A^{n-1} + \ldots + a_0 I = 0 \]

From this we obtain:

\[ e^{at} = \alpha_0(t)I + \alpha_1(t)A + \ldots + \alpha_{n-1}(t)A^{n-1} \]

2.3 Controllability

We briefly review some basic concepts and results for linear autonomous systems

\[ \dot{x} = Ax + Bu \]

\[ y = Cx \]

where \( x \in R^n, u \in R^m, y \in R^p \). Recall that given an initial state \( x(0) = x_0 \) and a control \( u(t), t > 0 \), the corresponding trajectory is defined by the variations of parameters formula

\[ x(t; x_0, u) = e^{At} x_0 + \int_{0}^{t} e^{A(t-s)} Bu(s) ds \]
**Definition:** A state \( x_1 \in \mathbb{R}^n \) is reachable from \( x_0 \) if there exists a finite time \( t > 0 \) and a piecewise continuous control \( u \) such that \( x(t;x_0,u) = x_1 \). \( \mathcal{R}_{x_0} \) denotes the set of states reachable from \( x_0 \).

Let us make a few preliminary observations.

If \( x_1 \) is reachable from \( x_0 \) in some time \( t_1 > 0 \), it is reachable in every time \( t \). To see this simply rescale \( s \):

\[
\int_0^{t} e^{A(t-s)} Bu(s)ds = \int_0^{t} e^{A(t-s)} Bu(s)ds = \int_0^{t} e^{A(t-s)} Bu(s)ds
\]

Thus, we have the replacement \( u(s) \to e^{A(t-t)} u(s) \).

Notice that \( x_1 \) is reachable from \( x_0 \) if and only if \( x_1 - e^{At} x_0 \) is reachable from the origin for \( 0 < t < \infty \), viz

\[
x_1 = e^{At} x_0 + \int_0^{t} e^{A(t-s)} Bu(s)ds \iff x_1 - e^{At} x_0 = \int_0^{t} e^{A(t-s)} Bu(s)ds
\]

As a result, we focus on characterizing the set of states reachable from the origin. Let \( \mathcal{U} \) denote the linear vector space of control functions \( u(\tau), \tau \in [0,t_1] \), and \( \mathcal{X} \equiv \mathbb{R}^n \) the space of states \( x(t_1) \). The inner product on \( \mathcal{U} \) is the usual

\[
\langle \hat{u}, u \rangle = \int_0^{t_1} \hat{u}(\tau)u(\tau)d\tau
\]

The mapping \( A: \mathcal{U} \to \mathcal{X} \) defined by

\[
x(t_1) = \int_0^{t_1} e^{A(t-t_1)} Bu(\tau)d\tau
\]

defines the state reached from the origin at time \( t_1 \) when the control \( u \) is applied on the time interval \([0,t_1]\).

A state \( x_1 \) is reachable from the origin over the time interval \([0,t_1]\) if and only if the relation \( A(u) = x_1 \) has a solution \( u(t) \). Such a solution exists if and only if \( x_1 \in \text{Im} A \). It is more convenient to apply the equivalent condition \( x_1 \in \text{Im} AA^* \) because \( AA^*: \mathcal{X} \to \mathcal{X} \).

So let’s us prove this result.
Lemma 1: There exists a solution \( u \) of \( A(u) = x_i \) if and only if \( x_i \in \text{Im} \ A^* A \).

Proof: Sufficiency \( (x_i \in \text{Im} \ A^* A \Rightarrow x_i \in \text{Im} \ A) \) is obvious. To prove necessity, assume \( x_i = A(u) \) and \( x_i \notin \text{Im} \ A^* A \). Since \( \mathcal{X} = \text{Im} \ A \oplus \text{ker} \ A^* \), \( x_i \) has a component in \( \text{ker} \ A^* \). It follows that there exists an \( x_2 \) such that \( A^* x_2 = 0 \) and \( x_i^T x_2 \neq 0 \). Now,

\[
x_2^T A^* x_2 = 0 \Rightarrow \| A^* x_2 \| = 0 \Rightarrow A^* x_2 = 0
\]

Then

\[
x_i^T x_2 = \langle Au, x_2 \rangle_{\mathcal{X}} = \langle u, A^* x_2 \rangle_{\mathcal{X}} = 0
\]

contradicting the condition \( x_i^T x_2 \neq 0 \).

Qed

To use this result, we need to calculate the adjoint mapping \( A^*: \mathcal{X} \rightarrow \mathcal{Y} \). It is defined by

\[
\langle x, Au \rangle_\mathcal{X} = \langle A^* x, u \rangle_\mathcal{Y}
\]

Thus,

\[
\int_0^t \left( A^*(x) \right)^T u(\tau) d\tau = x_i^T \int_0^t e^{A(\tau-s)} Bu(s) ds = \int_0^t \left( B^T e^{A^* (\tau-t)} x \right)^T u(\tau) d\tau
\]

From this we identify

\[
A^*(x) = B^T e^{A^* (t-t)} x
\]

so that

\[
AA^*(x) = \left[ \int_0^t e^{A(t-\tau)} BB^T e^{A^* (t-\tau)} d\tau \right] x
\]

Consequently, in view of Lemma 1, we have the following result.

Proposition: The set of states reachable from the origin over the time interval \([0, t_1]\) is

\[
\text{Im} \ A A^* = \text{Im} \left[ \int_0^t e^{A(t-\tau)} BB^T e^{A^* (t-\tau)} d\tau \right]
\]

Definition: The matrix

\[
G_c(t_1) = \int_0^t e^{A(t-\tau)} BB^T e^{A^* (t-\tau)} d\tau
\]
is called the controllability Gramian.

**Definition:** The system or the matrix pair \((A, B)\) is said to be (completely) controllable if any state \(x_i \in \mathbb{R}^n\) is reachable from any other state \(x_0 \in \mathbb{R}^n\) in finite time.

**Proposition:** The system is completely controllable if and only if

\[
\text{rank } G_c(t_i) = n
\]

Choose a point \(\eta \in \mathcal{X}\) and use it to construct a control \(u(t) = A^* (\eta) = B^T e^{A^T(t_i-t)} \eta\). Starting at \(x(0) = 0\), apply the control to obtain.

\[
x(t_i) = AA^* (\eta) = \left[ \int_0^{t_i} e^{A^T(\tau-t)} B B^T e^{A^T(\tau-t)} \ d\tau \right] \eta = G_c(t_i) \eta
\]

(1.3)

Any \(x_i\) reachable from the origin satisfies this relation for some \(\eta\). Hence, a control that steers the system from the origin to \(x_i\) on the interval \([0, t_i]\) is

\[
u(t) = A^* (\eta) = B^T e^{A^T(t_i-t)} \eta
\]

(1.4)

for any \(\eta\) that satisfies

\[
G_c(t_i) \eta = x_i
\]

(1.5)

If the system is completely controllable there is a unique \(\eta\), yielding

\[
u(t) = B^T e^{A^T(t_i-t)} G_c^{-1}(t_i)x_i
\]

(1.6)

It is interesting to note that if the system is unstable, the control at any time \(t\) becomes small as \(t_i \to \infty\). The opposite occurs if the system is stable. This is consistent with the well known fact that highly maneuverable vehicles tend to be unstable or marginally stable.

Let \(B := \text{Im}(B)\), and

\[
\langle A | B \rangle := B + AB + \cdots + A^{n-1}B = \text{Im} \left[ B \ AB \ \cdots A^{n-1}B \right]
\]

Then we have the following basic result

**Theorem:** \(R_0 = \langle A | B \rangle\).

Proof: We need to show that

\[
\langle A | B \rangle = \text{Im} \ G_c(t), \ t > 0
\]
Since $G_c(t)$ is symmetric ($\Rightarrow \mathcal{A} = \text{Im} G_c \oplus \ker G_c$), this is equivalent to

$$\left\langle A \mid B \right\rangle^\perp = \ker G_c(t), \ t > 0$$

First show, $x \in \ker G_c(t) \Rightarrow x \in \left\langle A \mid B \right\rangle^\perp$. If $x \in \ker G_c(t)$, then $x^T G_c x = 0$ so that

$$\int_0^t \| x^T e^{A(t-\tau)} B \|^2 d\tau = 0, \ 0 < \tau < t$$

Therefore

$$x^T e^{A_s} B = 0, \ 0 < s < t$$

Expanding $e^{A_s}$ and comparing coefficients leads to

$$x^T A^i B = 0, \ i = 0, \ldots, n - 1$$

Consequently, $x^T [B AB \ldots A^{n-1} B] = 0$ which implies that $x$ is orthogonal to $\left\langle A \mid B \right\rangle$, i.e.

$$x \in \left\langle A \mid B \right\rangle^\perp.$$ 

Now show $x \in \ker G_c(t) \Leftrightarrow x \in \left\langle A \mid B \right\rangle^\perp$. But $x \in \left\langle A \mid B \right\rangle^\perp$ implies $x^T [B AB \ldots A^{n-1} B] = 0$ so reversing the above steps leads to $x^T G_c x = 0$. This is true only if $x \in \ker G_c(t)$.

Qed

**Theorem:** The system or the matrix pair $(A, B)$ is (completely) controllable if and only if $R_0 = R^n$, or equivalently

$$\text{rank}[B AB \ldots A^{n-1} B] = n$$

Proof:

Qed

We wish to emphasize the geometric aspects of controllability and observability. To do so fully requires the concept of an invariant subspace.

**Definition:** A subspace $V \subseteq R^n$ is invariant with respect to $A$ if

$$A V \subseteq V$$

Clearly, every eigenvector defines a one-dimensional invariant subspace. Furthermore, the set of all vectors $h$ satisfying

$$Ah = \lambda h$$

is called the eigenspace of $A$ associated with the eigenvalue $\lambda$. Every eigenspace of $A$ is an invariant subspace as is every subspace that can be constructed as the sum of
eigenspaces. Perhaps less obvious is the fact that every invariant subspace is the direct sum of eigenspaces.

$R(0)$ is $A$-invariant, i.e., $AR(0) \subseteq R(0)$. In fact $R(0)$ is the smallest $A$-invariant subspace of $\mathbb{R}^n$ containing $B$. Moreover, if $\dim R(0) = n_1$ there exists a system of coordinates in which the state equations take the form:

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\
        x_2
\end{bmatrix} + \begin{bmatrix} B_1 \\
0
\end{bmatrix} u, \quad x_1 \in \mathbb{R}^{n_1}, \quad x_2 \in \mathbb{R}^{n-n_1}
\]

such that the pair $(A_{11},B_1)$ is completely controllable, i.e., $\langle A_{11} | B_1 \rangle = \mathbb{R}^n$, and in fact $x_1$ are coordinates in $R(0)$. Hence the restriction of the system to $R(0)$ ($x_2=0$) results in a controllable system. Thus, we refer to $R(0)$ as the controllable subspace.

**More on Invariance**

Recall that application of the linear control $u = Kx + v$, results in the closed loop system

\[
\dot{x} = (A + BK)x + Bv
\]

This motivates the following definition:

**Definition:** A subspace $V \subseteq \mathbb{R}^n$ is $(A,B)$–invariant if there exists a state feedback matrix $K$ such that

\[
(A + BK)V \subseteq V
\]

Now, the following theorem can be established.

**Theorem:** $V \subseteq \mathbb{R}^n$ is $(A,B)$–invariant if and only if

\[
AV \subseteq V + B
\]

### 2.4 Observability

We briefly review some basic concepts and results for linear autonomous systems

\[
\dot{x} = Ax + Bu
\]

\[
y = Cx
\]

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p$. Recall that given an initial state $x(0) = x_0$ and a control $u(t), t > 0$, the corresponding trajectory is define by the variations of parameters formula
\[ x(t;x_0,u) = e^{At}x_0 + \int_0^t e^{A(t-s)}Bu(s)ds \]

**Definition:** A state \(x_1 \in \mathbb{R}^n\) is indistinguishable from \(x_0\) if for every finite time \(t\) and piecewise continuous control \(u(t)\), \(y(t;x_1,u) = y(t;x_2,u)\). \(I(x_0)\) denotes the set of states indistinguishable from \(x_0\).

Define

\[ \mathcal{K} = \bigcap_{i=1}^n \ker \left( CA^{-1} \right) \]

**Theorem:**

\[ I(0) = \mathcal{K} \]

**proof:**

**Definition:** The system or the matrix pair \((C,A)\) is said to be (completely) observable if knowledge of \(u(t)\) and \(y(t)\) on a finite time interval determines the state trajectory on that interval.

**Theorem:** The system or the matrix pair \((C,A)\) is (completely) observable if and only if \(I(0) = \emptyset\).

**proof:** (Wonham)

\(I(0)\) is \(A\)-invariant. In fact \(I(0)\) is the largest \(A\)-invariant subspace of \(\mathbb{R}^n\) contained in \(\ker(C)\). If \(\dim(I(0)) = n_1\) there exists a set of coordinates in which the system equations are in the form

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
A_{11} & 0 \\
A_{21} & A_{22}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} +
\begin{bmatrix}
B_1 \\
B_2
\end{bmatrix} u,
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} \in \mathbb{R}^{n_1}, \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} \in \mathbb{R}^{n-n_1}
\]

\[ y = \begin{bmatrix} C_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]
such that the pair \((C_1,A_{11})\) is completely observable, and \(x_2\) are coordinates in \(I(0)\). We call \(I(0) = \emptyset\) the \textit{unobservable subspace}.

**More on Invariance**

Similar results to those established for controllability can be established for observability.

**Definition:** A subspace \(V \subseteq \mathbb{R}^n\) is \((C,A)\)-\textit{invariant} if there exists a matrix \(F\) such that

\[(A + FC)V \subseteq V\]

Note that \((A+FC)\) is the closed loop matrix resulting from output injection.

**Theorem:** \(V \subseteq \mathbb{R}^n\) is \((C,A)\)-invariant if and only if

\[A\left(V \cap \ker(C)\right) \subseteq V\]

**2.5 Kalm an Decomposition**

**2.6 Thorp-Morse Form**

**2.7 Zeros**

Recall for a linear system the output is related to the input and initial state by

\[Y(s) = C[sI - A]^{-1}x_0 + \left(C[sI - A]^{-1}B + D\right)U(s)\]

\[G(s) = \left(C[sI - A]^{-1}B + D\right) = k \frac{n(s)}{d(s)} = k \frac{(s-z_1)\cdots(s-z_m)}{(s-p_1)\cdots(s-p_n)}\]

\[Y(s) = \frac{CN(s)x_0}{d(s)} + k \frac{n(s)}{d(s)} \frac{1}{s-\lambda}\]

A partial fraction expansion of the second term shows that this equation can be written in the form

\[Y(s) = \left(\frac{CN(s)x_0}{d(s)} + \frac{\tilde{n}(s)}{d(s)}\right) + \frac{c_\lambda}{s-\lambda}, \text{ with } c_\lambda = G(\lambda)\]

Moreover, it can be shown that it is always possible to choose \(x_0\) so that the term in brackets vanishes. If \(x_0\) is so chosen, then

\[Y(s) = \frac{c_\lambda}{s-\lambda}\]
Moreover, if $\lambda$ is a system zero, $c_\lambda = G(\lambda) = 0$. Thus, $Y(s) = 0 \Rightarrow y(t) = 0$. In summary, if $\lambda$ is a system zero, there exists $x_0$ such that $x(t) = x_0$ and $u(t) = e^{\lambda t}$ results in $y(t) = 0$. This is the essential property of SISO system zeros that we intend to generalize.

Consider the MIMO system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Suppose $u(t) = ge^{\lambda t}$, $g \in R^n$. We ask if there exists a solution of the form $x(t) = x_0 e^{\lambda t}$, $x_0 \in R^n$ such that $y(t) \equiv 0$. The assumed solution must satisfy

$$\lambda x_0 e^{\lambda t} = Ax_0 e^{\lambda t} + Bge^{\lambda t}$$
$$0 = Cx_0 e^{\lambda t} + Dge^{\lambda t}$$

This leads to

$$- [\lambda I - A] x_0 + Bg = 0$$
$$Cx_0 + Dg = 0$$

or

$$\begin{bmatrix} \lambda I - A & B \\ -C & D \end{bmatrix} \begin{bmatrix} -x_0 \\ g \end{bmatrix} = 0$$

This represents $n + p$ equations in $n + m$ unknowns. Suppose

$$r = \text{rank} \begin{bmatrix} \lambda I - A & B \\ -C & D \end{bmatrix}$$

Let us consider the following cases.

$p < m$, equation (1.7) always has nontrivial solutions. If the system matrix has maximum rank, $r = n + p$, there are $m - p$ independent solutions.

$m < p$, and $r \geq n + m$ there are no nontrivial solutions of (1.7).

$r < n + \min(m, p)$ there are nontrivial solutions.
3 Nominal Controller Design: State Space Perspective

3.1 State Feedback

Pole Placement

The Linear Regulator Problem

3.2 Observers & the Separation Principle

3.3 Disturbance Rejection
4 Transfer Function Models

4.1 State Space to Transfer Function

4.2 Frequency Response

4.3 Poles & Zeros of Transfer Functions

4.4 Realizations
5 Closed Loop Transfer Functions

5.1 Well Posedness

5.2 Closed Loop Transfer Functions

\[ Y = G(U + D_1) \]
\[ U = K(R - Y + D_2) \]
\[ E = R - Y \]

**Output**

\[ Y(s) = \left[ I + G(s)K(s) \right]^{-1} R(s) - \left[ I + G(s)K(s) \right]^{-1} G(s)D_1(s) \]
\[ + \left[ I + G(s)K(s) \right]^{-1} G(s)K(s)D_2(s) \]

**Error**

\[ E(s) = \left[ I + G(s)K(s) \right]^{-1} R(s) - \left[ I + G(s)K(s) \right]^{-1} G(s)D_1(s) \]
\[ + \left[ I + G(s)K(s) \right]^{-1} G(s)K(s)D_2(s) \]

**Control**

\[ U(s) = K(s)\left[ I + G(s)K(s) \right]^{-1} R(s) - K(s)\left[ I + G(s)K(s) \right]^{-1} G(s)D_1(s) \]
\[ + K(s)\left[ I + G(s)K(s) \right]^{-1} \left\{ I + 2G(s)K(s) \right\} D_2(s) \]
### 6 Performance in the Frequency Domain

#### 6.1 Sensitivity Functions

\[
E(s) = \left[ I + L \right]^{-1} R(s) - \left[ I + L \right]^{-1} \left[ I + L \right]^{-1} \left[ I + L \right]^{-1} L D_2(s), \text{ where } L := GK
\]

Sensitivity function: \( S := \left[ I + L \right]^{-1} \)

Complementary sensitivity function: \( T := \left[ I + L \right]^{-1} L \)

Consider a scalar system in which \( L = GK \) is the open loop transfer function and \( T = [1 + L]^{-1} L \) is the closed loop transfer function. Then compute the (relative) variation of the closed loop with respect to (relative) variation of the open loop transfer function:

\[
\frac{dT/T}{dL/L} = \frac{dT}{dL} \frac{L}{T}
\]

\[
= \left\{ -[1 + L]^{-2} L + [1 + L]^{-1} \right\} \frac{L}{[1 + L]^{-1} L}
\]

\[
= -[1 + L]^{-1} L + 1
\]

\[
= [1 + L] = S
\]

This is Bode’s original reason for the terminology ‘sensitivity function’ for \( S \).

**A fundamental tradeoff**

Note that \( \left[ I + L \right]^{-1} + \left[ I + L \right]^{-1} L = I \)

\( S + T = I \)

\( T = \left[ I + L \right]^{-1} L = \left[ I + L^{-1} \right]^{-1} = L \left[ I + L \right]^{-1} \)

\( G \left[ I + KG \right]^{-1} = \left[ I + GK \right]^{-1} G \)

\( GK \left[ I + GK \right]^{-1} = G \left[ I + KG \right]^{-1} K = \left[ I + GK \right]^{-1} GK \)

\[
E(s) = S(s) R(s) - S(s) GD_1(s) + T(s) D_2(s)
\]

\[
U(s) = K(s) \left[ I + G(s) K(s) \right]^{-1} R(s) - K(s) \left[ I + G(s) K(s) \right]^{-1} G(s) D_1(s)
\]

\[
+ K(s) \left[ I + G(s) K(s) \right]^{-1} \left\{ I + 2G(s) K(s) \right\} D_2(s)
\]
6.2 Sensitivity Peaks

\[ M_S = \max_\omega |S(j\omega)|, \quad M_T = \max_\omega |T(j\omega)| \]

Sensitivity peaks are related to gain and phase margin.
Sensitivity peaks are related to overshoot and damping ratio.

6.3 Bandwidth

Bandwidth (sensitivity) \( \omega_{bs} = \max_v \left\{ v : |S(j\omega)| < 1/\sqrt{2} \quad \forall \omega \in [0,v) \right\} \)

Bandwidth (complementary sensitivity) \( \omega_{bt} = \min_v \left\{ v : |S(j\omega)| < 1/\sqrt{2} \quad \forall \omega \in (v,\infty) \right\} \)

Crossover frequency \( \omega_c = \max_v \left\{ v : |L(j\omega)| \geq 1 \quad \forall \omega \in [0,v) \right\} \)

Bandwidth is related to rise time and settling time.

6.4 Limits on Performance
7 Nominal Controller Design: Frequency Domain Perspective

7.1 Full State Feedback Controllers

The Quadratic Regulator Problem

\[ J(x,t) = x^T(T)Qx(T) \quad + \quad \int_0^T \left[ x^T(\tau)Qx(\tau) + u^T(\tau)Ru(\tau) \right] d\tau \]

Principle of Optimality & HJB Equation

Stability of Sol’ns to LQR

Consider the linear system \( \dot{x} = Ax \). Suppose \( V(x) = x^T Px \) with \( P > 0 \). Compute the time rate of change of \( V \) along trajectories

\[ \dot{V} = \frac{\partial V}{\partial x} \dot{x} = x^T PAx + x^T A^T Px \]  

(1.1)

Suppose \( \dot{V} = -x^T Qx, \quad Q > 0 \). Then, clearly, the system is asymptotically stable, \( x(t) \to 0 \) as \( t \to \infty \). Actually, if \( \det A \neq 0 \), we require only \( Q \geq 0 \). Hence, if there exists a \( P > 0 \) that satisfies the Liapunov equation

\[ PA + A^T P = -Q \]  

(1.2)

for any \( Q \geq 0 \), the system is asymptotically stable provided \( \det A \neq 0 \). As a matter of fact this requirement is necessary, as well as sufficient, for asymptotic stability.

Now, consider the linear control system

\[ \dot{x} = Ax + Bu \]  

(1.3)

Suppose that \( u = Kx \) so that the closed loop system is

\[ \dot{x} = (A + BK)x \]  

(1.4)

Moreover, choose \( K = -R^{-1}B^T Px \) where

\[ PA + A^T P - PBR^{-1}B^T P = -Q \]  

(1.5)

Now, (1.5) can be rewritten

\[ P(A + BK) + (A + BK)^T P = -Q - PBR^{-1}B^T P \]
A Generalization of the Standard LQR

Consider, the more general performance index:

$$J(x,t) = x^T(T)Q_f x(T) + \int_{0}^{T} \left[ x^T(\tau)Qx(\tau) + 2x^T(\tau)Su(t) + u^T(\tau)Ru(\tau) \right]d\tau$$

with $R > 0$, and $Q - SR^{-1}S \geq 0$. We can reduce this problem to the standard case by noting the identity (complete the square)

$$x^T Qx + 2x^T Su + u^T Ru = \left( u + R^{-1}Sx \right)^T R \left( u + R^{-1}Sx \right) + x^T \left( Q - SR^{-1}S^T \right) x$$

Define a new control

$$\tilde{u} = u + R^{-1}S^T x$$

so that the system equations become

$$\dot{x} = Ax + Bu \Rightarrow \dot{x} = \left( A - BR^{-1}S^T \right) x + B\tilde{u}$$

and in terms of the new control, the performance index is

$$J(x,t) = x^T(T)Q_f x(T) + \int_{0}^{T} \left[ x^T(\tau) \left( Q - SR^{-1}S^T \right) x(\tau) + \tilde{u}^T(\tau)R\tilde{u}(\tau) \right]d\tau$$

Thus, the problem has been recast in the form of the standard regulator problem. The solution is

$$u = -R^{-1} \left( B^T P + S \right) x$$

$$P \left( A - BR^{-1}S^T \right) + \left( A - BR^{-1}S^T \right)^T P - PB \left( P + \left( Q - SR^{-1}S^T \right) \right)$$

Min-Max Control

Consider a system by disturbances $w(t)$ with performance outputs $y(t)$

$$\dot{x} = Ax + Bu + Ew$$

$$y = Cx$$

(1.6)

The disturbance is norm bounded, but otherwise unknown. Our goal is to find a state feedback control that produces minimum quadratic cost for the ‘worst-case’ disturbance. To make this statement precise, we set up the performance index:

$$J(u,w) = \int_{0}^{\infty} \left[ y^T(t)y(t) + \rho u^T(t)u(t) - y^2w^T(t)w(t) \right] dt$$

(1.7)
where the weighting constants \( \rho, \gamma > 0 \). By explicitly including the disturbances in the cost in a negative way, stationary points of \( J(u, w) \) will be saddle points. We seek \( u(t) \) that minimizes \( J \) while a perverse nature seeks \( w(t) \) that maximizes it. The optimization problem is to find

\[
\min_u \max_w J(u, w)
\]

**Proposition:** Consider the system (1.6) with performance index (1.7). Suppose that
1. \( w(t) \) has bounded energy,
2. \( (A, B) \) and \( (A, E) \) are stabilizable,
3. \( (A, C) \) is detectable.

Then, if the optimal solution exists, it is a unique saddle point of \( J(u, w) \) where
1. The optimal min-max control is
   \[
u(t) = Kx(t), \quad K = -\frac{1}{\rho}B^T S\]
2. The worst case disturbance is
   \[
w(t) = \frac{1}{\gamma^2}E^T Sx(t)\]

\( S \) is the unique, symmetric, nonnegative solution of the algebraic Riccati equation

\[
A^T S + SA - S \left( \frac{1}{\rho}BB^T - \frac{1}{\gamma^2}EE^T \right) = -C^T C
\]

Recall that solution of the Riccati equation can be carried out by eigen-decomposition of its associated Hamiltonian matrix. As long as the Hamiltonian matrix has no eigenvalues on the imaginary axis, the required decomposition can be performed. The Hamiltonian matrix for the min-max optimization problem is

\[
H = \begin{bmatrix}
A & \frac{1}{\gamma^2}EE^T - \frac{1}{\rho}BB^T \\
-C^T C & -A^T
\end{bmatrix}
\]
Given the detectability/stabilizability assumptions it is possible to prove there exists a $\gamma_{\min}$ so that there are no eigenvalues of $H$ on the imaginary axis provided $\gamma > \gamma_{\min}$. In fact, when $\gamma \to \infty$ we approach the standard LQR solution. For $\gamma = \gamma_{\min}$, the controller is the full state feedback $H_\infty$ controller. All other values of $\gamma_{\min} \leq \gamma < \infty$ produce valid min-max controllers.

The condition that $\Re \lambda(H) \neq 0$ is equivalent to stability of the matrix

$$A + \frac{1}{\gamma^2} EE^T - \frac{1}{\rho} BB^T$$

Notice that the closed loop system matrix is

$$A - \frac{1}{\rho} BB^T$$

Since $EE^T / \gamma^2$ is destabilizing, the feedback system has some margin of stability.

**Solving the Riccati Equation**

**Constructing the Solution**

**Computing Tools**

CARE  Solve continuous-time algebraic Riccati equations.

\[
[X, L, G, RR] = \text{CARE}(A, B, Q, R, S, E) \quad \text{computes the unique symmetric stabilizing solution } X \text{ of the continuous-time algebraic Riccati equation}
\]

\[
-A'XE + E'XA - (E'XB + S)R - (B'XE + S') + Q = 0
\]

or, equivalently,

\[
-F'XE + E'XF - E'XBR - B'XE + Q - SR S' = 0 \quad \text{with } F := A - BR, S'.
\]

When omitted, $R, S$ and $E$ are set to the default values $R=I$, $S=0$, and $E=I$. Additional optional outputs include the gain matrix

\[
G = R^{-1} (B'XE + S')
\]

the vector $L$ of closed-loop eigenvalues (i.e., $\text{EIG}(A-B*G, E)$),
and the Frobenius norm RR of the relative residual matrix.

\[ [X, L, G, \text{REPORT}] = \text{CARE}(A, B, Q, \ldots, 'report') \]
turns off error messages and returns a success/failure diagnosis REPORT instead.
The value of REPORT is
* -1 if Hamiltonian matrix has eigenvalues too close to jω axis
* -2 if X=X2/X1 with X1 singular
* the relative residual RR when CARE succeeds.

\[ [X1, X2, L, \text{REPORT}] = \text{CARE}(A, B, Q, \ldots, 'implicit') \]
also turns off error messages, but now returns matrices X1, X2 such that X=X2/X1.
REPORT=0 indicates success.

### 7.2 Output Feedback Controllers

#### The Classical $H_2$ Problem – LQG
The classical output feedback optimal control problem for SISO systems was solved during the 2nd World War using a frequency domain formulation. It is referred to as the Wiener-Hopf-Kolmogorov problem. Attempts to extend this result to the MIMO case using frequency domain techniques were not fruitful. The MIMO problem was formulated and solved in the state space by Kalman and coworkers around 1960. We summarize the result here.

#### The Linear Quadratic Guassian (LQG) Problem - Setup
The standard problem formulation is as follows. The plant is described by
\[
\dot{x} = Ax + Bu + w \\
y = Cx + v
\]
The disturbances $w, v$ are independent, zero-mean white noise processes have covariances,
\[
E\{w(t)w^T(\tau)\} = W\delta(t-\tau) \\
E\{v(t)v^T(\tau)\} = V\delta(t-\tau)
\]
and
\[
E\{w(t)v^T(\tau)\} = 0
\]
We seek $u(t)$ that minimizes the performance index:

$$J = E\left\{ \lim_{T \to \infty} \frac{1}{T} \int_0^T x^T Q x + u^T R u \, dt \right\}, \quad Q = Q^T \geq 0, \ R = R^T > 0$$

**Solution Summary**

$$u = K\hat{x}(t), \ K = -R^{-1}B^T P$$

$$PA + A^T P - PB R^{-1} B^T P = -Q, \ P \geq 0$$

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}), \ L = SC^T V^{-1}$$

$$SA^T + AS - SC^T V^{-1} CS = -W, \ S \geq 0$$

**The Modern Paradigm**

We view the control design problem in terms of the diagram shown in Figure 1.

![Figure 1](image)

**Figure 1.** The so-called ‘modern paradigm’ views the plant in terms of two input sets: disturbance and control inputs, and two output sets: performance and measured variables.

In the frequency domain the plant is characterized in terms of a transfer matrix:

$$\begin{bmatrix} z \\ v \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}$$

Closing the loop with

$$u = K v$$

Produces the closed loop transfer function

$$z = F w, \ F = P_{11} + P_{12} (I - P_{22} K)^{-1} P_{21}$$
In the state space the plant is defined as follows in terms of differential equations

\[ \dot{x} = Ax + B_1w + B_2u \]
\[ z = C_1x + D_{11}w + D_{12}u \]
\[ y = C_2x + D_{21}w + D_{22}u \]

and, for convenience, we define the data structure

\[ P := \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} \]

In the following we will make the following standing assumptions about the plant:

1. \( D_{11} = 0 \)
2. \( (A, B_2) \) is stabilizable
3. \( (A, C_2) \) is detectable
4. \( V = \begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} \begin{bmatrix} B_1^T & D_{21}^T \end{bmatrix} := \begin{bmatrix} V_{xx} & V_{xy} \\ V_{yx} & V_{yy} \end{bmatrix} \geq 0 \) with \( V_{yy} > 0 \)
5. \( R = \begin{bmatrix} C_1^T \\ D_{12}^T \end{bmatrix} \begin{bmatrix} C_1 & D_{12} \end{bmatrix} := \begin{bmatrix} R_{xx} & R_{xu} \\ R_{ux} & R_{uu} \end{bmatrix} \geq 0 \) with \( R_{uu} > 0 \)

Our goal is to find an output \( (y) \) feedback controller that optimizes a performance measure defined in terms of the performance variables \( (z) \) for some specified class of disturbance inputs \( (w) \).
**Figure 2.** The standard LQG problem recast in the modern paradigm.

**Frequency Domain Formulation of the $H_2$ Problem**

The ‘energy norm’ or ‘2-norm’ of any scalar function $f(t)$ is defined by

$$
\|f(t)\|_2^2 := \int_{-\infty}^{\infty} f^2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(j\omega)|^2 d\omega
$$

The latter obtained from Parseval’s theorem. This is easily generalized to vector a function

$$
\|f(t)\|_2^2 = \int_{-\infty}^{\infty} f^T(t)F(T)dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} F^T(j\omega)F(j\omega)d\omega
$$

Now, let us consider the control system shown in the figure. Suppose, we consider the class of disturbances to be a zero mean white noise with

$$
E\left\{w^T(t)w(t+\tau)\right\} = I\delta(\tau)
$$

We seek to choose $K$ so that the expected value of $\|z(t)\|_2$ is a minimum. Thus, we seek to

$$
\min_K E\left\{\|z(t)\|_2^2\right\}
$$

Recall, $E[\delta(t)]=1$. Now, compute
\[ \int_{-\infty}^{\infty} z^T(t)z(t)\,dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} Z^T(-j\omega)Z(j\omega)\,d\omega \]
\[ = \text{tr}\left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} Z(j\omega)Z^T(-j\omega)\,d\omega \right\} \]

\[ E\left\{ \|z\|_2^2 \right\} = \text{tr}\left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} E\left\{ Z(j\omega)Z^T(-j\omega) \right\}d\omega \right\} \]
\[ = \text{tr}\left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega)F^T(-j\omega)d\omega \right\} \]
\[ = \left\| F \right\|_2^2 \]

**The H_2 Problem**

We consider the same control design problem as above, except with the class of disturbances defined by
\[ \|w(t)\|_2 = 1 \]

We seek to choose \( K \) such that the maximum of \( \|z(t)\|_2 \) over all disturbance inputs \( w(t) \) is a minimum. Thus, we seek to
\[ \min_{K} \max_{\|w\|_2} \|z(t)\|_2 \]

Now, compute
\[ \int_{-\infty}^{\infty} z^T(t)z(t)\,dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} Z^T(-j\omega)Z(j\omega)\,d\omega \]
\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} W^T(-j\omega)F^T(-j\omega)F(j\omega)W(j\omega)d\omega \]

Now, we seek the maximum performance energy over all disturbances with unit norm. It occurs when \( W(j\omega) \) is aligned with the maximum eigenvalue of \( F^*F \),
\[ \max_{\|w\|_2} \int_{-\infty}^{\infty} z^T(t)z(t)\,dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} W^T(j\omega)\bar{\sigma}^2(F(j\omega))W(j\omega)d\omega \]
\[ = \max_{\omega} \bar{\sigma}^2(F(j\omega)) = \left\| F \right\|_\infty^2 \]

**Solution Summary**
**Proposition (H₂ Output Feedback):** Suppose \( w(t) \) is a unit intensity white noise signal, \( E\{w(t)w^T(\tau)\} = I \delta(t-\tau) \). Then the unique, stabilizing, optimal controller that minimizes \( \|T_{zw}(s)\|_2 \) is

\[
K_2 = \begin{bmatrix} A_2 & -L_2 \\ F_2 & 0 \end{bmatrix}
\]

where

\[
A_2 = A + B_2F_2 + L_2C_2 + L_2D_2F_2
\]

\[
F_2 = -R^{-1}_{uu} \left( R^{-1}_{uu} + B_2^T X_2 \right)
\]

\[
L_2 = -\left( Y_2C^T_2 + V_{xy} \right) V^{-1}_{yy}
\]

\( X_2, Y_2 \) satisfy the two Riccati equations

\[
X_2 A_r + A_r^T X_2 - X_2 B_2 R^{-1}_{uu} B_2^T X_2 = -R_{xx} + R_{uu} R^{-1}_{uu} R_{uu}^T
\]

\[
A_r Y_2 + Y_2 A_r^T - Y_2 C_2^T V^{-1}_{yy} C_2 Y_2 = -V_{xx} + V_{xy} V^{-1}_{xy} V_{xy}^T
\]

\[
A_r = \left( A - B_2 R^{-1}_{uu} R_{uu}^T \right)
\]

\[
A_r = \left( A - V_{xy} V^{-1}_{yy} C_2 \right)
\]
Proposition (\( H_\infty \) Output Feedback): Suppose \( w(t) \) is a bounded \( L_2 \) signal, 
\[
\int_{-\infty}^{\infty} w^T(t)w(t)dt < \infty.
\]
A stabilizing controller that satisfies 
\[
\left\| T_{zw}(s) \right\|_\infty < \gamma
\]
is
\[
K_\infty = \begin{bmatrix} A_\infty & -Z_\infty L_\infty \\ F_\infty & 0 \end{bmatrix}
\]
where
\[
A_\infty = A + \left( B_1 + L_\infty D_{21} \right) W_\infty + B_2 F_\infty + Z_\infty L_\infty C_2 + Z_\infty L_\infty D_{22} F_\infty
\]
\[
F_\infty = -R^{-1}_{wu} \left( R^T_{wu} + B_2^T X_\infty \right)
\]
\[
L_\infty = -\left( Y_\infty C^T_2 + V_{yy} \right) V^{-1}_{yy}
\]
\[
W_\infty = \frac{1}{\gamma^2} B_1^T X_\infty, \quad Z_\infty = \left( I - \frac{1}{\gamma^2} Y_\infty X_\infty \right)
\]
\( X_\infty \geq 0 \) and \( Y_\infty \geq 0 \) satisfy the Riccati equations
\[
X_\infty A_\gamma + A_\gamma^T X_\infty - X_\infty \left( B_2 R^{-1}_{wu} B_2^T - \frac{1}{\gamma^2} B_2 B_1^T \right) X_\infty = -R_{xx} + R_{wu} R^{-1}_{wu} R_{wu}^T
\]
\[
A_\gamma Y_\infty + Y_\infty A_\gamma^T - Y_\infty \left( C_2^T V^{-1}_{yy} C_2 - \frac{1}{\gamma^2} C_1^T C_1 \right) Y_\infty = -V_{xx} + V_{xy} V^{-1}_{yy} V_{xy}^T
\]
and the following conditions are satisfied:

1. The Hamiltonian matrix
\[
\begin{bmatrix}
A - B_2 R^{-1}_{wu} R_{wu}^T & -B_2 R^{-1}_{wu} B_2^T + \frac{1}{\gamma^2} B_2 B_1^T \\
-R_{xx} + R_{wu} R^{-1}_{wu} R_{wu}^T & \left( A - B_2 R^{-1}_{wu} R_{wu}^T \right)^T
\end{bmatrix}
\]
has no eigenvalues on the imaginary axis, or equivalently,
\[
A + B_1 W_\infty + B_2 F_\infty
\]
is stable.

2. The Hamiltonian matrix
\[
\begin{bmatrix}
\left( A - V_{xy} V^{-1}_{yy} C_2 \right)^T & -C_2^T V^{-1}_{yy} C_2 + \frac{1}{\gamma^2} C_1^T C_1 \\
-V_{xx} + V_{xy} V^{-1}_{yy} V_{xy}^T & -A + V_{xy} V^{-1}_{yy} C_2
\end{bmatrix}
\]
has no eigenvalues on the imaginary axis, or equivalently,
\[ A + L_\omega C_2 + \frac{1}{\gamma^2} Y_\omega C_1^T C_1 \]

is stable.

3. \( \rho(Y_\omega X_\omega) < \gamma^2 \), where \( \rho(\cdot) = \max_i |\lambda_i(\cdot)| \) is the spectral radius.
8 Robust Stability & Nyquist Analysis

8.1 SISO Nyquist Analysis
Guaranteed Gain and Phase Margin

**Proposition:** Suppose $M_s$ is the maximum sensitivity peak. Then

$$GM \geq \frac{1}{1 \pm \alpha}, \quad PM \geq \pm 2 \sin \left( \frac{\alpha}{2} \right), \quad \alpha = 1/M_s$$

**Proof:** The proof is easily obtained from the geometry shown in the figure.

8.2 MIMO Nyquist Analysis

Recall that the closed loop poles are roots of the polynomial

$$d(s) = d_e(s) \det[I + L(s)]$$

If the open loop system is stable, then to assess stability of the closed loop system we need only be concerned with the zeros of

$$F(s) = \det[I + L(s)]$$

Nyquist analysis still applies with $\det[I + L(s)]$ replacing $1 + L(s)$. Specifically we have:

$$Z = P - N$$

where:

- $Z$ = number of closed loop poles in the RHP
- $P$ = number of open loop poles in the RHP
- $N$ = number of counterclockwise encirclements of the $F$-plane origin by $F(C)$.
Suppose that the closed loop system is asymptotically stable. Then the Nyquist
\[ \det \{ I + L(s) \} \neq 0, \ \forall s = \sigma + j \omega, \sigma \geq 0 \]

### 8.3 $M \Delta$-Structure

Models of Uncertainty
\[
\begin{align*}
G_p &= G + E \\
G_p &= G(I + E) \\
G_p &= (I + E)G \\
G_p &= G(I - EG)^{-1}, \quad E = W_1 \Delta W_2, \quad \| \Delta \|_\infty < 1 \\
G_p &= G(I - E)^{-1} \\
G_p &= (I - E)^{-1}G
\end{align*}
\]
\[
M = W_1 M_0 W_2, \\
M_0 &= K(I + GK)^{-1} = KS \\
M_0 &= K(I + GK)^{-1}G = T_i \\
M_0 &= GK(I + GK)^{-1} = T \\
M_0 &= (I + GK)^{-1}G = SG \\
M_0 &= (I + GK)^{-1} = S_i \\
M_0 &= (I + GK)^{-1} = S
\]

### 8.4 Small Gain Theorem

Consider a feedback loop with open loop transfer matrix $L(s)$. Then we define the spectral radius:
\[
\rho(L(j\omega)) = \max_i |\lambda_i(L(j\omega))| 
\]

**Theorem** (Spectral radius stability theorem). Consider a system with stable open loop transfer function $L(s)$. Then the closed loop is stable if
\[
\rho(L(j\omega)) < 1, \ \forall \omega
\]

**Theorem** (Small gain theorem). Consider a system with stable open loop transfer function $L(s)$. Then the closed loop is stable if
\[
\|L(j\omega)\| < 1, \ \forall \omega
\]

where $\|L\|$ denotes any norm satisfying $\|AB\| \leq \|A\| \cdot \|B\|$. 
8.5 Robust Stability of the $M$ $\Delta$ - Structure

**Theorem** (Robust stability for unstructured perturbations). Assume that the nominal system $M(s)$ is stable and that the perturbations $\Delta(s)$ are stable. Then the $M\Delta$ -system is stable for all perturbations satisfying $\|\Delta\|_\infty \leq 1$ if and only if

$$\bar{\sigma}(M(j\omega)) < 1, \ \forall \omega \ \iff \ \|M\|_\infty < 1$$

Notice that:

$$\text{RS} \iff \|M\|_\infty < 1 \iff \|W_1 M_0 W_2\|_\infty < 1$$

Special case

$$G_p = G(I + w\Delta), \|\Delta\|_\infty < 1 \iff \|wT\|_\infty < 1$$
9 Robust Performance
10 Some Complex Variable Concepts

For proper, stable, minimum phase systems: gain $\Leftrightarrow$ phase:

$$H(0) > 0$$

10.1 Analytic Functions

A function $f(s)$ of a complex variable $s$ is analytic at $s_0 \in C$ if it is differentiable on a neighborhood of $s_0$.

Example:

$$H(s) = K \frac{(s - q_1) \cdots (s - q_m)}{(s - p_1) \cdots (s - p_n)} = \frac{n(s)}{d(s)}$$

is analytic everywhere except at its poles. Notice that $\ln H = \ln n(s) - \ln d(s)$, so

$$\frac{d \ln H}{ds} = \frac{1}{n(s)} \frac{d n(s)}{ds} - \frac{1}{d(s)} \frac{d d(s)}{ds}$$

hence $\ln H$ is analytic everywhere except at poles and zeros of $H$.

Cauchy Integral Theorem:

Suppose that $f$ is analytic on a domain $D$, and suppose $\partial \Omega$ is any piecewise smooth simple closed curve in $D$, then

$$\oint_{\partial \Omega} f(s) ds = 0$$

Cauchy Integral Formula:

Suppose that $\partial \Omega$ is a simple closed curve $\partial \Omega$ in the complex plane, and $f$ is an analytic function on $\partial \Omega$ and its interior $\Omega$. Then

$$f(s) = \frac{1}{2\pi i} \oint_{\partial \Omega} \frac{f(z)}{z-s} dz, \text{ for all } s \in \Omega.$$ 

Residue Theorem

Suppose $f(s)$ is an analytic function having isolated singular points. Let $C$ be a simple closed curve that encloses $n$ singular points having residues $c_i, i = 1, \ldots, n$. Then

$$\oint_{C} f(s) ds = 2\pi i \sum_{i=1}^{n} c_i$$
Poisson Integrals

Various integral formulas can be derived from the Cauchy integral formula by choosing specific contours. Consider the contour shown, with \( R \to \infty \). The integral will exist only if the function \( f(s) \) has restricted behavior at infinity. In particular, we require

\[
\lim_{R \to \infty} \frac{m(R)}{R} = 0, \quad \text{where} \quad m(R) := \sup_{\theta} \left| f(Re^{i\theta}) \right|, \quad \theta \in [-\pi/2, \pi/2]
\]

If \( f(s) \) has no singularities in RHP two equivalent formulas can be derived:

\[
f(s) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sigma f(jv)}{\sigma^2 + (\omega - v)^2} \, dv, \quad s = \sigma + j\omega
\]

\[
f(s) = \frac{1}{\pi j} \int_{-\infty}^{\infty} \frac{(\omega - v)f(jv)}{\sigma^2 + (\omega - v)^2} \, dv
\]

The last equation can be broken down into real and imaginary parts to yield:

\[
\text{Re} \, f(s) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(\omega - v)\text{Im} \, f(jv)}{\sigma^2 + (\omega - v)^2} \, dv, \quad \text{Im} \, f(s) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(\omega - v)\text{Re} \, f(jv)}{\sigma^2 + (\omega - v)^2} \, dv
\]

Thus \( \text{Re} \, f(j\omega) \Rightarrow \text{Im} \, f(s) \). However, we can not evaluate \( f(s) \) on the imaginary axis by simply setting \( \sigma = 0 \), because the integrals do not exist since the integrands have a pole at \( v = \omega \). However, a careful limiting process leads to
\[ \text{Im } f(j\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d}{d\alpha} \log \coth \left( \frac{\alpha}{2} \right) \, d\alpha \]

Suppose \( H(s) \) is a transfer function. Then consider

\[ f(s) = \log H(s) \]

for which \( \text{Re } f(s) = |H(s)| \) and \( \text{Im } f(s) = \angle H(s) \). Then

\[ \angle H(j\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d}{d\alpha} \log |H(j\omega)| \, \log \coth \left( \frac{\alpha}{2} \right) \, d\alpha \]

Since \( f(s) \) must be analytic in RHP, \( H(s) \) can have neither poles nor zeros in RHP.

**Bode Waterbed Formula**

Application of the Cauchy Integral Formula to systems with relative degree 2 or greater:

(Waterbed effect)

\[ \int_0^{\infty} \ln |S(j\omega)| \, d\omega = \pi \sum_{\text{ORHP poles}} p_i \]

\[ \int_0^{\infty} \ln |T(j\omega)| \, \frac{d\omega}{\omega^2} = \pi \sum_{\text{ORHP zeros}} \frac{1}{q_i} \]

**Example:** Stable plant

\[ L(s) = \frac{1}{(s + 1)^2} \]
10.2 Parseval's Theorem

A simple, but important formula is given by Parseval’s theorem. Suppose the functions $f_1(t), f_2(t)$ have Laplace transforms $F_1(s), F_2(s)$, respectively. Then we can write

$$
\int_{-\infty}^{\infty} f_1(t)f_2(t)dt = \int_{-\infty}^{\infty} L^{-1}[F_1(s)]f_2(t)dt
$$

$$
= \frac{1}{2\pi j} \int_{-\infty}^{\infty} \int_{-j\infty}^{j\infty} F_1(s)e^{st}ds f_2(t)dt
$$

$$
= \frac{1}{2\pi j} \int_{-\infty}^{\infty} \int_{-j\infty}^{j\infty} f_2(t)e^{st}dt F_1(s)ds
$$

$$
= \frac{1}{2\pi j} \int_{-\infty}^{\infty} F_2(-s)F_1(s)ds
$$

$$
\int_{-\infty}^{\infty} f^2(t)dt = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} F(-s)F(s)ds
$$

$$
= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(-j\omega)F(j\omega)d\omega
$$

$$
= \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(j\omega)|^2 d\omega
$$
11 Normed Linear Spaces

11.1 Norms and Normed Linear Spaces

Definition: A linear space over the field \( R \) (or \( C \)) is a set of elements \( \{x, y, \ldots\} \) such that

For each pair \( x, y \in \mathbb{R} \), the sum is defined \( x + y \in \mathbb{R} \) and \( x + y = y + x \)

there is an element \( 0 \in \mathbb{R} \) such that for every \( x \in \mathbb{R} \), \( x + 0 = x \)

for any numbers \( a, b \in R \) (or \( C \)) scalar multiplication is defined, \( ax \in \mathbb{R} \), and \( 1 \cdot x = x \), \( (ab)x = a(bx) = b(ax) \), and \( (a + b)x = ax + bx \) for all \( x, y \in \mathbb{R} \).

A linear space is a normed linear space if to each element \( x \in \mathbb{R} \) there corresponds a real number \( \|x\| \) called the norm of \( x \) which satisfies:

\[ \|x\| > 0 \text{ for } x \neq 0, \|0\| = 0 \]
\[ \|x + y\| \leq \|x\| + \|y\| \text{ (triangle inequality)} \]
\[ \|ax\| = |a| \cdot \|x\| \text{ for all } a \in R \text{ (or } C \text{) and } x \in \mathbb{R} \]

Examples of normed linear spaces:

1. The spaces \( R \) and \( C \) with \( \|x\| = |x| \), the absolute value of \( x \).

2. The real and complex Euclidean spaces \( R^n \) and \( C^n \) the spaces of scalar real or complex \( n \)-tuples with norm: \( \|x\| = \left( \sum_{i=1}^{n} |x_i|^2 \right)^{1/2} \)

3. The spaces \( l_p(n) \), space \( R^n \) or \( C^n \) with norm: \( \|x\| = \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p} \)

4. The space \( l_\infty(n) \), space \( R^n \) or \( C^n \) with norm: \( \|x\| = \max \{|x_1|,\ldots,|x_n|\} \). The notation is adopted because

\[ \max \{|x_1|,\ldots,|x_n|\} = \lim_{p \to \infty} \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p} \]

Prove this for the case \( n = 2 \).
5. The function spaces $L_p[a,b], 1 \leq p \leq \infty$ consisting of (complex-valued) integrable functions $f(t), t \in [a,b]$ with bounded norm: $\|f\|_p = \left[ \int_a^b |f(t)|^p \, dt \right]^{1/p}, \ 1 \leq p < \infty$ and $\|f\|_\infty = \sup_{t \in [a,b]} |f(t)|$.

6. The set of complex valued $m \times n$ matrices, denoted $C^{m \times n}$, with norm

$$\|A\| = \left[ \sum_{i,j} |a_{ij}|^2 \right]^{1/2} = \sqrt{\text{trace}(A^*A)} = \sqrt{\sum_{i=1}^n \sigma_i(A)},$$

or the norm $\|A\| = \sigma(A)$.

This is sometimes called the Frobenius norm.

7. Time-Domain (Signal) spaces. Consider complex vector valued time functions $f(t) \in \mathbb{R}^n$ defined on the interval $t \in (-\infty, \infty)$ (or, $t \in [0, \infty)$). The appropriate $p$-norm is

$$\|f\|_p = \left( \int_{-\infty}^{\infty} \left[ \sum_{i=1}^n |f_i(t)|^p \right]^{1/p} \, dt \right), \ 1 \leq p < \infty \text{ and } \|f\|_\infty = \sup_{t} \left( \max_i |f_i(t)| \right).$$

For $p = 2$ we can use Parseval’s theorem to obtain

$$\|f\|_2 = \left( \int_{-\infty}^{\infty} \left[ \sum_{i=1}^n |f_i(t)|^2 \right]^{1/2} \, dt \right) = \left( \int_{-\infty}^{\infty} \left[ \sum_{i=1}^n |F_i(j\omega)|^2 \right]^{1/2} \, d\omega \right) = \left( \int_{-\infty}^{\infty} F^* F \, d\omega \right)^{1/2}.$$

8. Frequency-Domain Spaces. Consider the functions $F(j\omega) \in \mathbb{C}^n$, $-\infty < \omega < \infty$

$$\|F\|_p = \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \sum_{i=1}^n \left| F_i(j\omega) \right|^p \right]^{1/p} \, d\omega \right), \ 1 \leq p < \infty \text{ and } \|F\|_\infty = \sup_{\omega} \left( \left| F^*(j\omega)F(j\omega) \right| \right).$$

The space of all frequency functions with bounded $p$-norm is often designated $L_p$.

9. The space of functions $F(s), F: \mathbb{C} \to \mathbb{C}^n$ that are analytic in the open right half plane, $\text{Re}(s) > 0$ with

$$\|F\|_p = \sup_{\xi > 0} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \sum_{i=1}^n \left| F_i(\xi + j\omega) \right|^p \right]^{1/p} \, d\omega \right), \ 1 \leq p < \infty \text{ and }$$

$$\|F\|_\infty = \sup_{\xi > 0} \left( \left| F^*(s)F(s) \right| \right).$$

The space of functions with bound $p$-norm is the Hardy space $H_p$. 
For almost all $\omega$, $\lim_{\xi \to 0} F(\xi + j\omega) = F(j\omega) \in L_p$. Moreover, the supremum occurs on the imaginary axis.

### 11.2 System Norms/ Induced Norms

Consider a mapping $A$ from one normed linear space $\square$ to another $\square$. That is, $y = Ax$.

One way to view the ‘magnitude’ of the action $A$ is consider its ‘gain,’ that is the ratio: $\|y\| / \|x\|$. The ratio will depend on the specific choice of input $x$. So, we choose the one that produces the largest ratio, i.e.:

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \max_{x \neq 0} \frac{\|Ax\|_\phi}{\|x\|_\phi}$$

Because $A$ is linear, it is clear that the gain is independent of the size of $x$ so we could rescale and equivalently define

$$\|A\| = \max_{\|x\| = 1} \|Ax\|_\phi$$

Now, the definition depends on the specific choices of input and output spaces $\square$ and $\square$.

Suppose that the $p$-norm is appropriate for both spaces, then we can define

$$\|A\|_{p\beta} = \max_{x \neq 0} \frac{\|Ax\|_\beta}{\|x\|_\alpha}$$

These induced norms have the product property:

$$\|A \circ B\| \leq \|A\||B\|$$

#### Transfer matrix norms

Consider systems described by $l \times m$ stable, proper transfer matrices:

$Y(s) = G(s)U(s)$

We will consider two transfer function norms:

**$H_2$ Norm**

$$\|G(s)\|_2 = \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{tr} \left[ G^*(j\omega)G(j\omega) \right] d\omega \right)^{1/2} = \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{j=1}^{\text{rank}(G)} \sigma_j[G(j\omega)] d\omega \right)^{1/2}$$

**$H_\infty$ Norm**
\[ \|G(s)\|_{\infty} = \sup_{\omega} \sigma[G(j\omega)] \]

Suppose that the input space is equipped with the \( L_2 \) norm and the output space with the \( L_\infty \). Then we have

\[ \|Y\|_{\infty} = \sup_{\omega} \left\{ \sqrt{U^* G^* GU} \right\} \]

so that

\[ \|G\|_2 = \max_{\|U\|_1} \sup_{\omega} \left\{ \sqrt{U^* G^* GU} \right\} \]

Suppose that the input and output spaces are equipped with the \( L_2 \) norm. Then we have

\[ \|Y\|_2^2 = U^* G^* GU \]

so that

\[ \|G\|_2 = \max_{\|U\|_1} \int_{-\infty}^{\infty} U^* (j\omega) G^* (j\omega) G(j\omega) U(j\omega) d\omega \]

We need to choose \( U \) to maximize the result. Clearly, at each \( \omega \) we maximize the kernel by aligning \( U \) with the largest eigenvector of \( G^* G \) so that we have

\[ U^* G^* GU(j\omega) = \sigma(\omega) \|U(j\omega)\|_2 \]

where \( \sigma(G(j\omega)) \) is the maximum singular value of \( G \).

Then we need to concentrate all of the energy in \( U \) at the frequency at which \( \sigma \) is a maximum

\[ \|G\|_\infty = \sup_{\omega} \sigma(G(j\omega)) \]

**Computing the \( H_2 \) Norm**

\[ AL_c + L_c A^T + BB^T = 0, \quad A^T L_o + L_o A + C^T C = 0 \]

\[ \|G\|_2 = \left[ \text{tr}(CL_c C^T) \right]^{1/2} = \left[ \text{tr}(B^T L_c B) \right]^{1/2} \]

**Computing the \( H_\infty \) Norm**

Find smallest \( \gamma > 0 \) such that \( H \) has no eigenvalues on the imaginary axis.

\[ H = \begin{bmatrix} A & \frac{1}{\gamma^2} BB^T \\ -C^T C & -A^T \end{bmatrix} \]