Robust Stability

MEM 355 Performance Enhancement of Dynamical Systems

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Outline

- Stability & Robustness
- Introduction – role of sensitivity functions
- Nyquist Analysis
- Traditional gain/phase margins
Introduction to Nyquist Stability Analysis

• Nyquist Analysis
  • A graphical method to determine how many closed loop poles are in the right half plane
  • Developed in the early 1930’s – those days it was not easy to find the roots of high order polynomials

• Stability Margins
  • Nyquist analysis provides a clear concept of ‘stability margin’
  • This concept generalizes to more complex MIMO systems
  • It remains a key concept in the current era
Cauchy Theorem

Theorem (Cauchy): Let $C$ be a simple closed curve in the $s$-plane. $F(s)$ is a rational function, having neither poles nor zeros on $C$. If $C_1$ is the image of $C$ under the map $F(s)$, then

$$Z = P - N$$

where

- $N$ the number of counterclockwise encirclements of the origin by $C_1$ as $s$ traverses $C$ once in the clockwise direction.
- $Z$ the number of zeros of $F(s)$ enclosed by $C$, counting multiplicities.
- $P$ the number of poles of $F(s)$ enclosed by $C$, counting multiplicities.
Nyquist

- Take $F(s) = 1 + L(s)$ (note: $F = S^{-1}$)
- Choose a $C$ that encloses the entire RHP
- Map into $L$-plane instead of $F$-plane (shift by -1)
Nyquist Theorem

**Theorem (Nyquist):** If the plot of $L(s)$ (i.e., the image of the Nyquist contour in the $L$-plane) encircles the point $-1+j0$ in the counterclockwise direction as many times as there are unstable open loop poles (poles of $L(s)$ within the Nyquist contour) then the feedback system has no poles in the RHP.

$$Z = P - N$$

closed loop poles in RHP = open loop poles in RHP - cc encirclements of -1
Example 1

\[ L(s) = \frac{1}{(s + p_1)(s^2 + 2\zeta\omega_0 s + \omega_0^2)} \]

I: \( s = j\omega \Rightarrow L(j\omega) = \frac{1}{(j\omega + p_1)(-\omega^2 + 2\zeta\omega_0 j\omega + \omega_0^2)}, 0 < \omega < \infty \)

II: \( s = \rho e^{j\theta}, \rho \to \infty, = \frac{\pi}{2} \downarrow -\frac{\pi}{2} \)

\[ L(\rho e^{j\theta}) = \frac{1}{(\rho e^{j\theta} + p_1)(\rho^2 e^{j2\theta} + 2\zeta\omega \rho e^{j\theta} + \omega_0^2)} \stackrel{\rho \to \infty}{\to} 0 \]

III: \( s = -j\omega, \ III \to I^* \)
Example 2

\[ L(s) = \frac{1}{s(s^2 + 2\zeta \omega_0 s + \omega_0^2)} \]

I: \( s = j\omega \Rightarrow L(j\omega) = \frac{1}{j\omega(-\omega^2 + j2\zeta \omega_0 \omega + \omega_0^2)}, 0 < \omega < \infty \)

II: \( s = \rho e^{j\theta}, \rho \rightarrow \infty, \theta = \frac{\pi}{2} \downarrow -\frac{\pi}{2} \)

\[ L(\rho e^{j\theta}) = \frac{1}{\rho e^{j\theta}\left(\rho^2 e^{j2\theta} + 2\zeta \omega_0 \rho e^{j\theta} + \omega_0^2\right)} \xrightarrow{\rho \rightarrow \infty} 0 \]

III: \( s = -j\omega, \quad III \rightarrow I^* \)

IV: \( s = \varepsilon e^{j\theta}, \varepsilon \rightarrow 0, \theta = -\frac{\pi}{2} \uparrow \frac{\pi}{2} \)

\[ L(\varepsilon e^{j\theta}) = \frac{1}{\varepsilon e^{j\theta}\left(\varepsilon^2 e^{j2\theta} + 2\zeta \omega_0 \varepsilon e^{j\theta} + \omega_0^2\right)} \xrightarrow{\varepsilon \rightarrow 0} \frac{1}{\varepsilon \omega_0^2} e^{-j\theta} \]
Example 3

\[
G(s) = \frac{1}{s(s^2 + 2(0.1)s + 1)}
\]

\[
\gg s=\text{tf('s')};
\gg G=1/(s*(s^2+2*0.1*s+1));
\gg \text{nyquist(G)}
\]

To obtain global picture
Gain & Phase Margin

Assumption: the nominal system is stable.

| $L(j\omega)$ | $0 < \omega < \infty$ |

Nyquist plot

Bode plot

Gain margin

Phase margin

$\gamma_m$

$\phi_m$

$|L(j\omega)|$

$\angle L(j\omega)$

Gain, db

Phase, deg

0.01 0.1 1 10

frequency
Robustness From Sensitivity Functions

Sensitivity peaks are related to gain and phase margin. Sensitivity peaks are related to overshoot and damping ratio.

\[ M_S = \max_\omega |S(j\omega)| \]
\[ M_T = \max_\omega |T(j\omega)| \]

\[ S = \rho e^{j\theta} = (1 + L)^{-1} \]
\[ \Rightarrow L = -1 + \rho^{-1} e^{-j\theta} \]
constant \(|S|\) prescribes circle

Unit circle centered at -1

As \(a\) shrinks we approach instability

\[ |S| = \frac{1}{a} \]

Principle part of Nyquist

Sensitivity peak

| \(L\) - plane | \(|S| > 1\) | \(|S| < 1\) |
| --- | --- | --- |
| \(S^{-1} = 1 + L\) | Im | Re |
Example: XV-15

Sensitivity function plots for $K=1, 5, 25$

Larger sensitivity peak, closer to instability, reduced gain and phase margins, reduced damping, increased overshoot
Example: single unstable pole

\[ L(s) = \frac{1}{(s - p_1)(s^2 + 2p\omega_0 + \omega_0^2)} \]
Example Cont’d- MATLAB Computations

\[ L(s) = \frac{0.6}{(s - 0.5)(s^2 + s + 1)} \]

```matlab
s = tf('s');
L = 0.60/((s-0.5)*(s^2 + s + 1));
figure
cyquist(L,1.25*L,0.835*L)
grid
```

- 0.835L(s) in orange: low gain limit
- L(s) in blue: stable
- 1.25L(s) in red: high gain limit
Summary

• Need to consider 2-3 transfer functions to fully evaluate performance
• Bandwidth is inversely related to settling time
• Sensitivity function peak is related to overshoot and inversely to damping ratio
• Gain and phase margins can be determined from Nyquist or Bode plots
• Sensitivity peak is inversely related to stability margin
• Design tools:
  • Bode and/or Nyquist diagrams helps establish robustness (margins) & performance (sensitivity peaks)