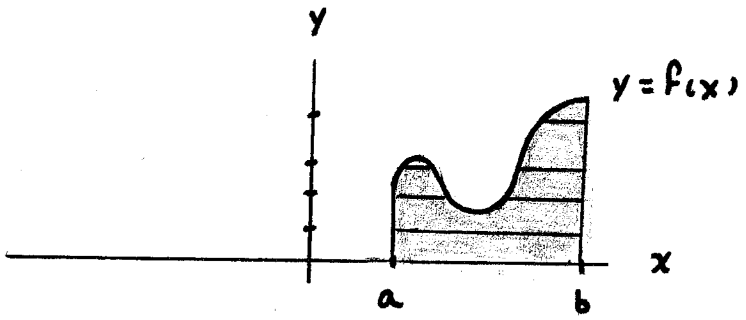
VOLUMES II

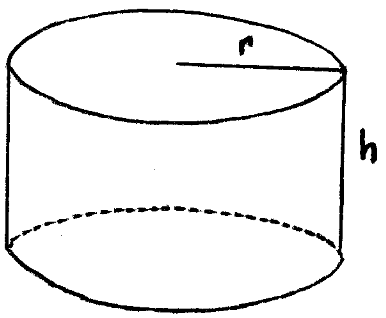
HERE IS ANOTHER VOLUME PROBLEM :

FIND THE VOLUME OF THE SOLID OBTAINED BY REVOLVING THE REGION



ABOUT THE y-AXIS. INTEGRATING CROSS-SECTIONAL AREAS WOULD BE PRETTY ANNOYING THIS TIME SINCE THE CROSS-SECTIONS ARE OF 4 DIFFERENT TYPES.

TO DESCRIBE AN ALTERNATIVE, RECALL THE FOLLOWING :



SURFACE AREA $= 2\pi r h$

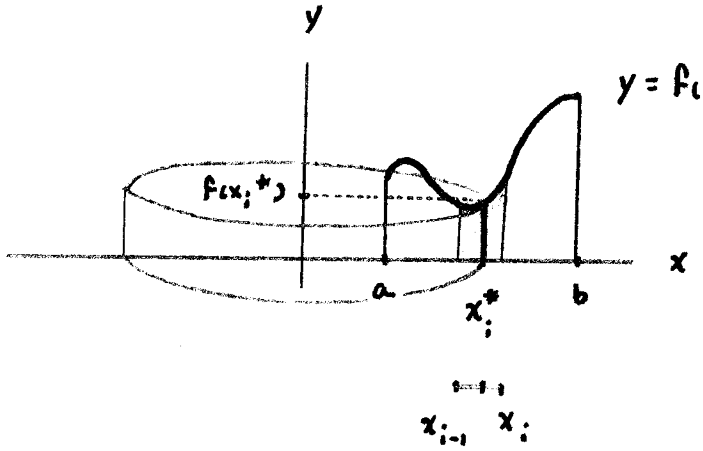
(SLICE AND FLATTEN)

NOW " THICKEN " THE CYLINDER BY Δr.

VOLUME $= 2\pi r h \Delta r$

(SLICE AND FLATTEN)

NOW RETURN TO THE SOLID DESCRIBED ABOVE :

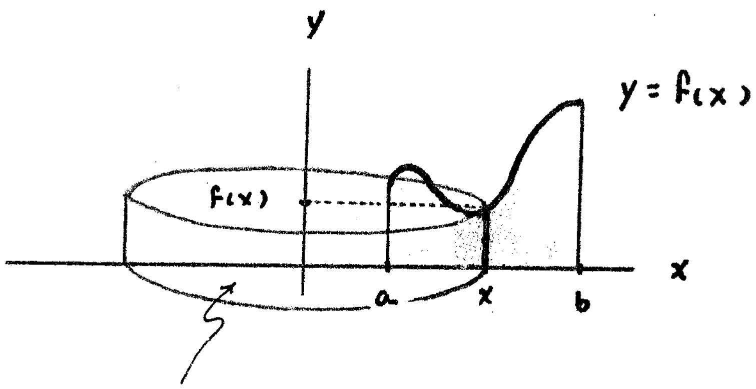


$$\text{VOLUME} \approx \sum_{i=1}^{n} 2\pi x_i^* f(x_i^*) \Delta x_i$$

$$\text{VOLUME} = \lim_{\Delta x_{max} \to 0} \sum_{i=1}^{n} 2\pi x_i^* f(x_i^*) \Delta x_i$$

$$= \int_a^b 2\pi x f(x) dx$$

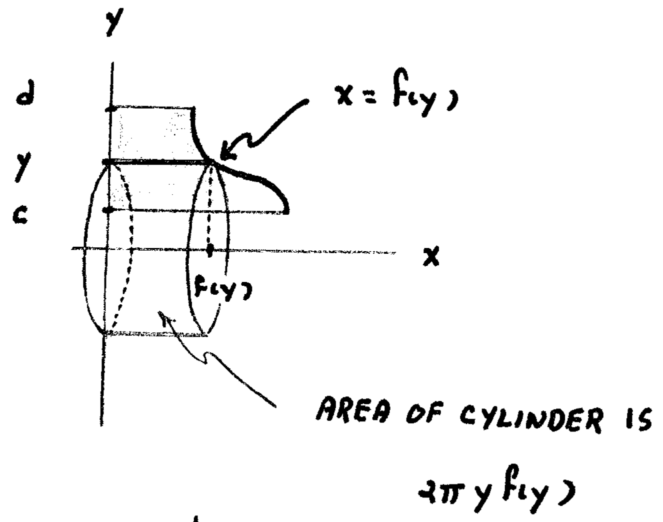
THE BEST WAY TO REMEMBER AND USE THIS FORMULA (AND ITS VARIANTS) IS AS FOLLOWS :



AREA OF CYLINDER IS $2\pi x f(x)$

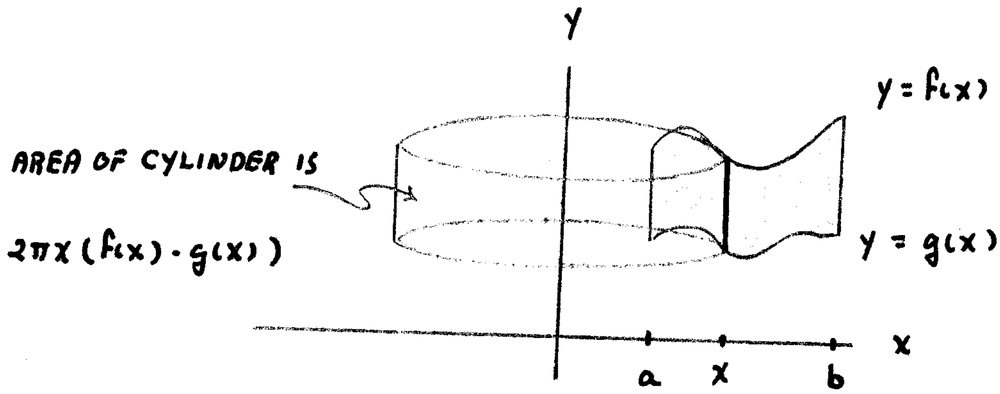
INTEGRATE THIS OVER $[a,b]$

SAME FOR REVOLVING ABOUT X-AXIS :



$$\text{VOLUME} = \int_c^d 2\pi y f(y)\,dy$$

SAME FOR THE REGION BETWEEN TWO GRAPHS, E.G.,

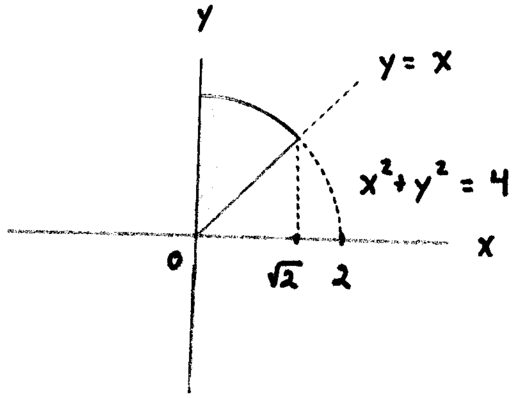


$$\text{VOLUME} = \int_a^b 2\pi x\,(f(x) - g(x))\,dx$$

AND SIMILARLY FOR REVOLVING ABOUT THE X-AXIS.

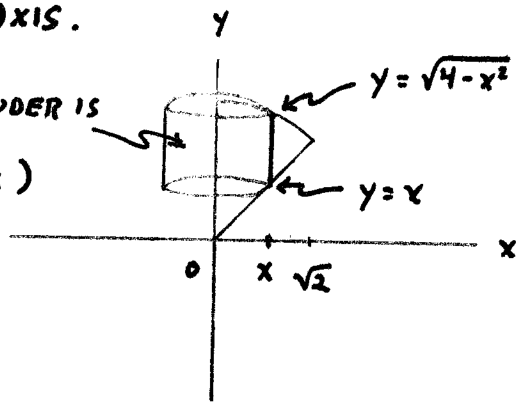
EXAMPLES :

1. CONSIDER THE REGION IN THE FIRST QUADRANT BOUNDED BY THE GRAPHS OF $x^2+y^2=4$ AND $y=x$.



FIND THE VOLUME OF THE SOLID OBTAINED BY REVOLVING THIS REGION ABOUT THE y-AXIS.

AREA OF CYLINDER IS $2\pi x\left(\sqrt{4-x^2}-x\right)$



$$\text{VOLUME} = \int_0^{\sqrt{2}} 2\pi x\left(\sqrt{4-x^2}-x\right)dx$$

$$= 2\pi\int_0^{\sqrt{2}} x(4-x^2)^{\frac{1}{2}}dx - 2\pi\int_0^{\sqrt{2}} x^2 dx$$

$$u = 4-x^2$$
$$du = -2x\,dx$$
$$x=0 \Rightarrow u=4$$
$$x=\sqrt{2} \Rightarrow u=2$$

$$\text{VOLUME} = -\pi \int^{\sqrt{2}} (4-x^2)^{\frac{1}{2}}(-2x\,dx) - 2\pi \left[\tfrac{1}{3}x^3\right]_0^{\sqrt{2}}$$

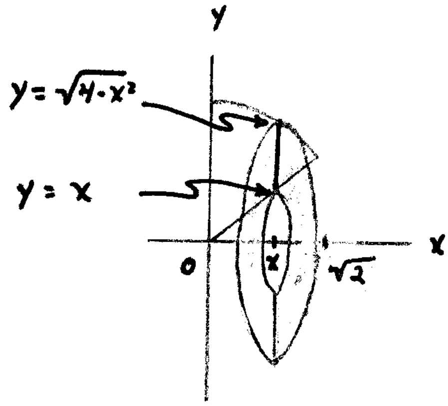
$$= -\pi \int_4^2 u^{\frac{1}{2}}du - 2\pi \left(\frac{2\sqrt{2}}{3} - 0\right)$$

$$= -\pi \left[\tfrac{2}{3}u^{\frac{3}{2}}\right]_4^2 - \frac{4\sqrt{2}\,\pi}{3}$$

$$= -\frac{2\pi}{3}\left[2\sqrt{2} - 8\right] - \frac{4\sqrt{2}\,\pi}{3} = -\frac{2\pi}{3}\left[2\sqrt{2} - 8 + 2\sqrt{2}\right]$$

$$= -\frac{2\pi}{3}\left[4\sqrt{2} - 8\right] = \frac{8\pi}{3}\left[2 - \sqrt{2}\right]$$

2. REVOLVE THE SAME REGION AS IN EXAMPLE 1 ABOUT THE X-AXIS AND FIND THE VOLUME OF THE RESULTING SOLID.



NOTICE THAT IN THIS CASE IT IS MORE SENSIBLE TO GO BACK TO INTEGRATING CROSS-SECTIONAL AREAS

$$\text{VOLUME} = \int_0^{\sqrt{2}} A(x)dx = \int_0^{\sqrt{2}} \left[\pi(\sqrt{4-x^2})^2 - \pi x^2\right]dx$$

$$= \pi\int_0^{\sqrt{2}} \left[4 - 2x^2\right]dx = \pi\left[4x\Big|_0^{\sqrt{2}} - \tfrac{2}{3}x^3\Big|_0^{\sqrt{2}}\right]$$

$$= \pi\left[4\sqrt{2} - \tfrac{2}{3}(2\sqrt{2})\right] = 4\sqrt{2}\,\pi\left[1 - \tfrac{1}{3}\right] = \frac{8\sqrt{2}\,\pi}{3}$$