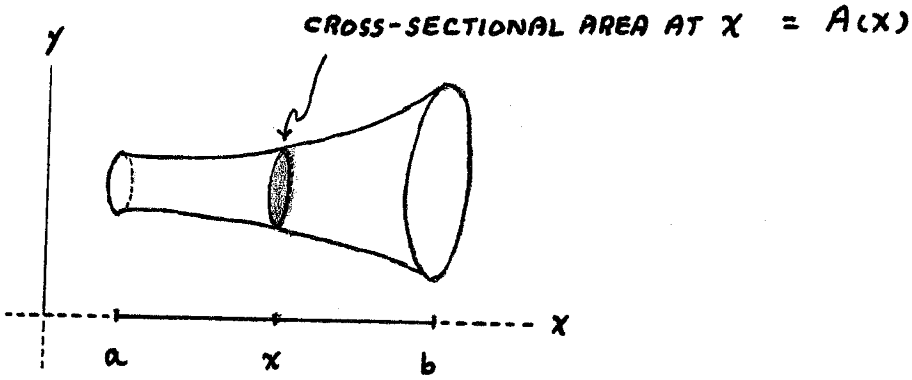
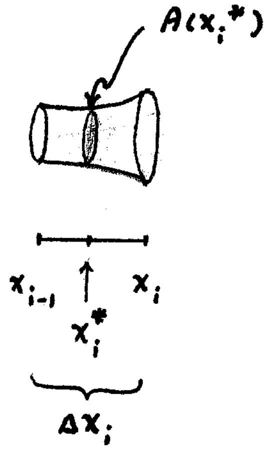
VOLUMES I

THE VOLUME OF A SOLID CAN BE COMPUTED BY "INTEGRATING ITS CROSS-SECTIONAL AREA"



$$\text{VOLUME} = \int_a^b A(x)\,dx$$

HERE'S THE REASON : <u>APPROXIMATE</u> THE VOLUME BY CARVING IT INTO THIN CYLINDRICAL SLABS



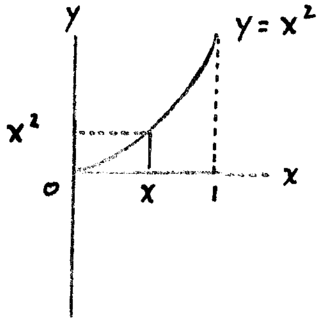
$$\text{VOLUME} \approx A(x_i^*)\Delta x_i$$

ADD $\sum_{i=1}^{n} A(x_i^*)\Delta x_i$. APPROXIMATION IMPROVES AS SLABS GET THINNER SO

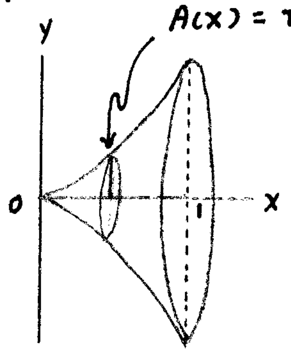
$$\text{VOLUME} = \lim_{\Delta x_{max} \to 0} \sum_{i=1}^{n} A(x_i^*)\Delta x_i = \int_a^b A(x)\,dx$$

EXAMPLES :

1. THE REGION UNDER THE GRAPH OF $y = f(x) = x^2$ AND ABOVE THE INTERVAL $[0,1]$ IS REVOLVED AROUND THE x-AXIS. COMPUTE THE VOLUME OF THE RESULTING SOLID.

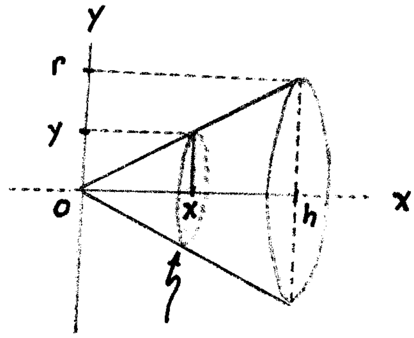


$A(x) = \pi r^2 = \pi (x^2)^2 = \pi x^4$



$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi x^4\,dx$$
$$= \pi \left(\tfrac{1}{5} x^5 \Big|_0^1 \right) = \frac{\pi}{5}$$

2. FIND THE VOLUME OF A CONE OF HEIGHT h AND RADIUS r.



SIMILAR TRIANGLES :

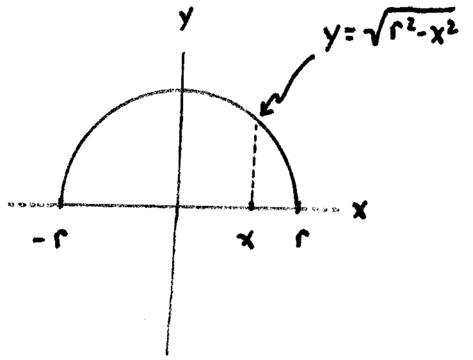
$$\frac{y}{x} = \frac{r}{h}$$
$$y = \frac{r}{h} x$$

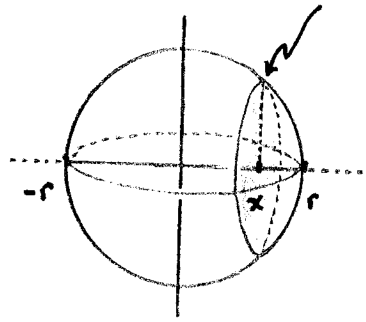
$$A(x) = \pi r^2$$
$$= \pi y^2 = \pi \left(\frac{r}{h} x \right)^2$$
$$= \frac{\pi r^2}{h^2} x^2$$

$$V = \int_0^h A(x)\,dx = \frac{\pi r^2}{h^2} \int_0^h x^2\,dx = \frac{\pi r^2}{h^2} \frac{1}{3} x^3 \Big|_0^h = \frac{1}{3} \pi r^2 h$$

3. FIND THE VOLUME OF A SPHERE OF RADIUS r.

THINK OF THE SPHERE AS THE SOLID OBTAINED BY REVOLVING THE REGION UNDER $y = f(x) = \sqrt{r^2 - x^2}$ AND ABOVE THE INTERVAL $[-r, r]$ ABOUT THE X-AXIS.





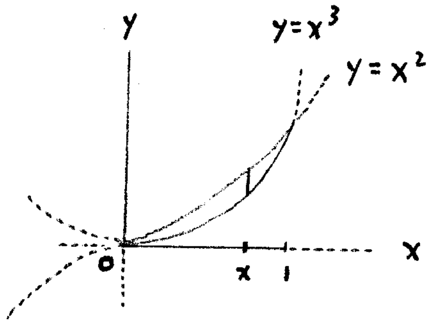
$A(x) = \pi r^2$
$= \pi(\sqrt{r^2-x^2})^2$
$= \pi(r^2 - x^2)$

$$V = \int_{-r}^{r} A(x)dx = \int_{-r}^{r} \pi(r^2-x^2)dx = \pi r^2 x \Big|_{-r}^{r} - \frac{\pi}{3}x^3\Big|_{-r}^{r}$$

$$= \pi r^2(r-(-r)) - \frac{\pi}{3}(r^3-(-r)^3) = 2\pi r^3 - \frac{2\pi}{3}r^3$$

$$= \frac{4}{3}\pi r^3$$

4. REVOLVE THE REGION BETWEEN THE GRAPHS OF $y = x^2$ AND $y = x^3$ ABOUT THE X-AXIS. COMPUTE THE VOLUME OF THE RESULTING SOLID.



INTERSECTIONS:

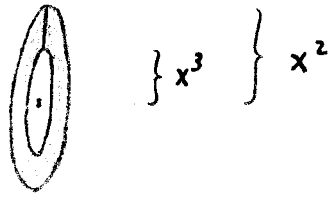
$x^3 = x^2$

$x^3 - x^2 = 0$

$x^2(x-1) = 0$

$x = 0, 1$

THIS TIME THE CROSS-SECTION AT x ISN'T A CIRCLE, BUT A WASHER (ANNULUS).



$$A(x) = (\text{AREA OF LARGE CIRCLE}) - (\text{AREA OF SMALL CIRCLE})$$

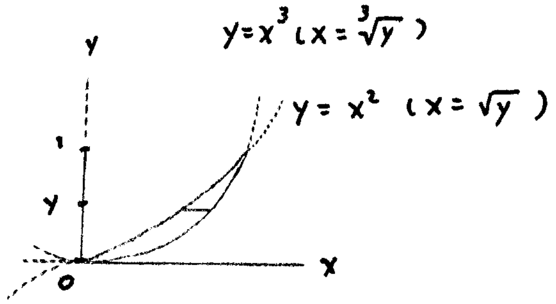
$$= \pi (x^2)^2 - \pi (x^3)^2$$

$$= \pi x^4 - \pi x^6$$

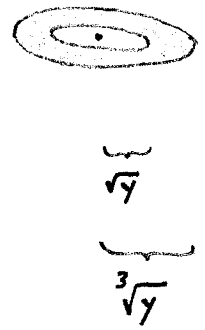
$$V = \int_0^1 A(x)\,dx = \pi \int_0^1 (x^4 - x^6)\,dx = \pi \left[\tfrac{1}{5} x^5 \Big|_0^1 - \tfrac{1}{7} x^7 \Big|_0^1 \right]$$

$$= \pi \left[\tfrac{1}{5} - \tfrac{1}{7} \right] = \frac{2\pi}{35}$$

5. THIS TIME WE'LL REVOLVE THE REGION IN EXAMPLE #4 ABOUT THE y-AXIS TO PRODUCE A SOLID AND COMPUTE ITS VOLUME.



JUST REVERSE THE ROLES OF THE VARIABLES. FOR EACH y BETWEEN 0 AND 1 THE CROSS-SECTION IS AN ANNULUS

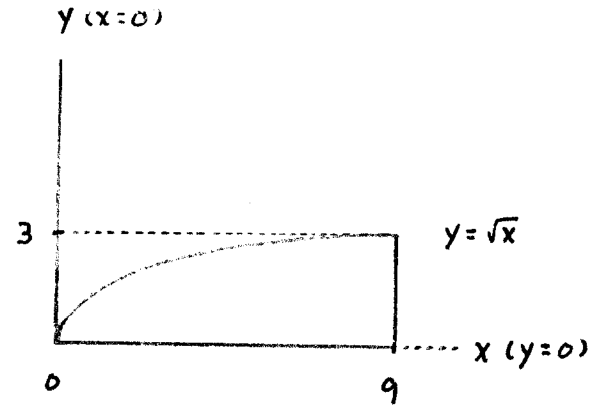


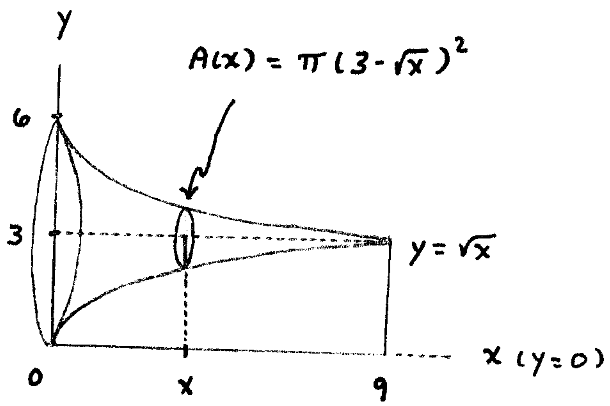
$$A(y) = \pi(\sqrt[3]{y})^2 - \pi(\sqrt{y})^2 = \pi y^{\frac{2}{3}} - \pi y$$

$$V = \int_0^1 A(y)\,dy = \pi\int_0^1 (y^{\frac{2}{3}} - y)\,dy = \pi\left[\tfrac{3}{5} y^{\frac{5}{3}}\Big|_0^1 - \tfrac{1}{2}y^2\Big|_0^1\right]$$

$$= \pi\left[\tfrac{3}{5} - \tfrac{1}{2}\right] = \frac{\pi}{10}$$

6. FIND THE VOLUME OF THE SOLID OBTAINED BY REVOLVING THE REGION ENCLOSED BY $y = \sqrt{x}$, $y = 3$ AND $x = 0$ ABOUT THE LINE $y = 3$.





$$V = \int_0^9 A(x)\,dx = \pi\int_0^9 (3-\sqrt{x})^2\,dx = \pi\int_0^9 (9 - 6\sqrt{x} + x)\,dx$$

$$= \pi\left[9x\Big|_0^9 - 4x^{\frac{3}{2}}\Big|_0^9 + \tfrac{1}{2}x^2\Big|_0^9\right] = \frac{27}{2}$$