

## TRIGONOMETRIC SUBSTITUTION

TRIGONOMETRIC SUBSTITUTION : USEFUL IDENTITIES :

$$1 - \sin^2 \theta = \cos^2 \theta$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$\sec^2 \theta - 1 = \tan^2 \theta$$

OR, MORE GENERALLY,

$$a^2 - (a \sin \theta)^2 = (a \cos \theta)^2$$

$$a^2 + (a \tan \theta)^2 = (a \sec \theta)^2$$

$$(a \sec \theta)^2 - a^2 = (a \tan \theta)^2$$

NOW CONSIDER AN INTEGRAL SUCH AS

$$\int \frac{1}{x^2 \sqrt{4-x^2}} dx$$

NOTE :  $-2 < x < 2$

THE IDEA IS TO INTRODUCE A NEW VARIABLE  $\theta$  THAT WILL TURN  $4-x^2$  INTO A PERFECT SQUARE, THUS GETTING RID OF THE SQUARE ROOT.

WANT :  $4 - x^2 = ( )^2$

$2^2 - x^2 = ( )^2$

LOOKS LIKE :

$2^2 - (2 \sin \theta)^2 = (2 \cos \theta)^2$

SUGGESTS CHOOSING

$x = 2 \sin \theta$

THEN

$$\begin{aligned} \sqrt{4-x^2} &= \sqrt{2^2-x^2} = \sqrt{2^2-(2\sin\theta)^2} \\ &= \sqrt{(2\cos\theta)^2} \\ &= 2\cos\theta \end{aligned}$$

NOTE : MUST ASSUME  
 $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$dx = 2 \cos \theta d\theta$

THUS,

$$\begin{aligned} \int \frac{1}{x^2 \sqrt{4-x^2}} dx &= \int \frac{1}{(2\sin\theta)^2 (2\cos\theta)} (2\cos\theta d\theta) \\ &= \frac{1}{4} \int \csc^2 \theta d\theta \\ &= -\frac{1}{4} \cot \theta + C \\ &= -\frac{1}{4} \frac{\cos \theta}{\sin \theta} + C \\ &= -\frac{1}{4} \frac{\frac{1}{2} \sqrt{4-x^2}}{\frac{1}{2} x} + C = -\frac{\sqrt{4-x^2}}{4x} + C \end{aligned}$$

MORE EXAMPLES :

$$1. \int \frac{dx}{\sqrt{1+x^2}}$$

NOTE : IF YOU HAPPEN TO REMEMBER THE INVERSE HYPERBOLIC FUNCTIONS, THEN YOU KNOW THAT

$$\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1} x + C = \ln |x + \sqrt{x^2+1}| + C$$

IF YOU DON'T, THEN DO A TRIGONOMETRIC SUBSTITUTION AS FOLLOWS :

$$1 + x^2 = (\quad)^2$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$x = \tan \theta$$

$$\sqrt{1+x^2} = \sec \theta$$

NOTE :  $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$dx = \sec^2 \theta d\theta$$

$$\int \frac{dx}{\sqrt{1+x^2}} = \int \frac{\sec^2 \theta d\theta}{\sec \theta} = \int \sec \theta d\theta =$$

$$\ln |\sec \theta + \tan \theta| + C = \ln |\sqrt{1+x^2} + x| + C$$

$$2. \int \frac{\sqrt{x^2 - 25}}{x} dx$$

NOTE: ASSUME  $x > 5$

$$x^2 - 5^2 = ( )^2$$

$$(5 \sec \theta)^2 - 5^2 = (5 \tan \theta)^2$$

$$x = 5 \sec \theta$$

$$\sqrt{x^2 - 25} = 5 \tan \theta$$

NOTE: ASSUME  $0 \leq \theta < \frac{\pi}{2}$

$$dx = 5 \sec \theta \tan \theta d\theta$$

$$\int \frac{\sqrt{x^2 - 25}}{x} dx = \int \frac{5 \tan \theta}{5 \sec \theta} (5 \sec \theta \tan \theta) d\theta =$$

$$5 \int \tan^2 \theta d\theta = 5 \int (\sec^2 \theta - 1) d\theta = 5 \tan \theta - 5\theta + C =$$

$$\sqrt{x^2 - 25} - 5 \operatorname{Arctan} \left( \frac{\sqrt{x^2 - 25}}{5} \right) + C$$

NOTE: COULD WRITE  $\theta = \operatorname{Arcsec} \frac{x}{5}$

BUT IT IS GENERALLY BEST TO STICK

WITH THE INVERSES OF SIN, COS AND TAN.

$$3. \int_5^{10} \frac{\sqrt{x^2 - 25}}{x} dx$$

THERE ARE TWO OPTIONS FOR DOING A DEFINITE INTEGRAL:

A. INDEFINITE INTEGRAL FIRST AND USE ORIGINAL LIMITS:

$$\int \frac{\sqrt{x^2-25}}{x} dx = \dots = \sqrt{x^2-25} - 5 \operatorname{ARCTAN}\left(\frac{\sqrt{x^2-25}}{5}\right) + C$$

so

$$\begin{aligned} \int_5^{10} \frac{\sqrt{x^2-25}}{x} dx &= \left. \sqrt{x^2-25} - 5 \operatorname{ARCTAN}\left(\frac{\sqrt{x^2-25}}{5}\right) \right|_5^{10} \\ &= (5\sqrt{3} - 0) - 5 (\operatorname{ARCTAN}(\sqrt{3}) - \operatorname{ARCTAN} 0) \\ &= 5\sqrt{3} - 5 \left( \frac{\pi}{3} - 0 \right) \\ &= 5\sqrt{3} - \frac{5\pi}{3} \end{aligned}$$

B. CHANGE THE LIMITS WHILE MAKING THE SUBSTITUTION:

$$\begin{aligned} \int_5^{10} \frac{\sqrt{x^2-25}}{x} dx &= 5 \int_0^{\frac{\pi}{3}} \tan^2 \theta d\theta = \dots \\ &= 5 \tan \theta \Big|_0^{\frac{\pi}{3}} - 5 \theta \Big|_0^{\frac{\pi}{3}} \\ x = 5 \sec \theta & \\ x = 5 \Rightarrow \sec \theta = 1 & \\ \Rightarrow \theta = 0 & \\ x = 10 \Rightarrow \sec \theta = 2 & \\ \Rightarrow \cos \theta = \frac{1}{2} & \\ \Rightarrow \theta = \frac{\pi}{3} & \end{aligned}$$

$$4. \int \frac{x^2}{\sqrt{9+x^2}} dx$$

$$9 + x^2 = ( )^2$$

$$3^2 + x^2 = ( )^2$$

$$3^2 + (3 \tan \theta)^2 = (3 \sec \theta)^2$$

$$x = 3 \tan \theta$$

$$\sqrt{9+x^2} = 3 \sec \theta \quad \text{NOTE: ASSUME } -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$dx = 3 \sec^2 \theta d\theta$$

$$\int \frac{x^2}{\sqrt{9+x^2}} dx = \int \frac{9 \tan^2 \theta}{3 \sec \theta} (3 \sec^2 \theta d\theta) = 9 \int \tan^2 \theta \sec \theta d\theta =$$

$$9 \int (\sec^2 \theta - 1) \sec \theta d\theta = 9 \int (\sec^3 \theta - \sec \theta) d\theta$$

REDUCTION FORMULA:

$$\int \sec^3 \theta d\theta = \frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \int \sec \theta d\theta$$

$$= \frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln |\sec \theta + \tan \theta| + C$$

so

$$\int \frac{x^2}{\sqrt{9+x^2}} dx = 9 \int \sec^3 \theta d\theta - 9 \int \sec \theta d\theta$$

$$= \frac{9}{2} \sec \theta \tan \theta + \frac{9}{2} \ln |\sec \theta + \tan \theta|$$

$$- 9 \ln |\sec \theta + \tan \theta| + C$$

$$= \frac{9}{2} \sec \theta \tan \theta - \frac{9}{2} \ln |\sec \theta + \tan \theta| + C$$

$$= \frac{9}{2} \left( \frac{\sqrt{9+x^2}}{3} \right) \left( \frac{x}{3} \right) - \frac{9}{2} \ln \left| \frac{\sqrt{9+x^2}}{3} + \frac{x}{3} \right| + C$$

$$= \frac{1}{2} x \sqrt{9+x^2} - \frac{9}{2} \ln \left| \sqrt{9+x^2} + x \right| + C$$

QUESTION: WHERE'D THE 3 GO?

5.  $\int \frac{1}{x^2 - 4x + 5} dx$

NOTE: NO SIMPLE SUBSTITUTION WILL WORK AND THE DENOMINATOR

DOES NOT FACTOR. NEW IDEA: COMPLETE THE SQUARE

$$x^2 - 4x + 5 = (x^2 - 4x + \underline{\quad}) + 5 - \underline{\quad}$$

$\uparrow$  ADD THE SQUARE OF HALF THE COEFFICIENT OF X
  $\uparrow$  AND SUBTRACT IT HERE

$$= (x^2 - 4x + 4) + 5 - 4$$

$$= (x-2)^2 + 1$$

$$\int \frac{1}{x^2 - 4x + 5} dx = \int \frac{1}{(x-2)^2 + 1} dx = \int \frac{1}{u^2 + 1} du =$$

$$u = x - 2$$

$$du = dx$$

$$\arctan u + C = \arctan (x-2) + C$$

$$6. \int \frac{x}{x^2 - 4x + 8} dx =$$

$$x^2 - 4x + 8 = (x^2 - 4x + 4) + 8 - 4 = (x-2)^2 + 4$$

$$\int \frac{x}{(x-2)^2 + 4} dx = \int \frac{u+2}{u^2+4} du = \int \left( \frac{u}{u^2+4} + \frac{2}{u^2+4} \right) du =$$

$$u = x-2 \quad (x = u+2)$$

$$du = dx$$

$$\int \frac{u}{u^2+4} du + 2 \int \frac{1}{u^2+4} du = \frac{1}{2} \int \frac{1}{u^2+4} (2u du) + 2 \int \frac{1}{4 \left( \left( \frac{u}{2} \right)^2 + 1 \right)} du$$

$$= \frac{1}{2} \ln |u^2+4| + \frac{1}{2} \int \frac{1}{\left( \frac{u}{2} \right)^2 + 1} du$$

$$= \frac{1}{2} \ln (u^2+4) + \frac{1}{2} \cdot 2 \int \frac{1}{\left( \frac{u}{2} \right)^2 + 1} \left( \frac{1}{2} du \right)$$

$$= \frac{1}{2} \ln (u^2+4) + \text{arctan} \left( \frac{u}{2} \right) + C$$

$$= \frac{1}{2} \ln ((x-2)^2+4) + \text{arctan} \left( \frac{x-2}{2} \right) + C$$

$$= \frac{1}{2} \ln (x^2 - 4x + 8) + \text{arctan} \left( \frac{x-2}{2} \right) + C$$

$$7. \int \frac{dx}{\sqrt{5-4x-2x^2}}$$

$$5-4x-2x^2 = -2x^2 - 4x + 5 = -2(x^2 + 2x \quad ) + 5$$

$$= -2(x^2 + 2x + 1) + 5 + 2$$

↑  
 ADDING 1 INSIDE THE  
 PARENTHESES REALLY  
 ADDS A -2.

$$\begin{aligned}
 5 - 4x - 2x^2 &= -2(x+1)^2 + 7 \\
 &= 7 - 2(x+1)^2 = 7 - (\sqrt{2}(x+1))^2 \\
 &= 7 \left( 1 - \left( \frac{\sqrt{2}(x+1)}{\sqrt{7}} \right)^2 \right)
 \end{aligned}$$

$$\int \frac{1}{\sqrt{5-4x-2x^2}} dx = \int \frac{1}{\sqrt{7} \sqrt{1 - \left( \frac{\sqrt{2}(x+1)}{\sqrt{7}} \right)^2}} dx =$$

$$\frac{1}{\sqrt{7}} \int \frac{1}{\sqrt{1 - \left( \frac{\sqrt{2}(x+1)}{\sqrt{7}} \right)^2}} dx = \frac{1}{\sqrt{7}} \frac{\sqrt{7}}{\sqrt{2}} \int \frac{1}{\sqrt{1-u^2}} du =$$

$$u = \frac{\sqrt{2}(x+1)}{\sqrt{7}}$$

$$du = \frac{\sqrt{2}}{\sqrt{7}} dx$$

$$\frac{1}{\sqrt{2}} \text{ARCSIN } u + C = \frac{1}{\sqrt{2}} \text{ARCSIN} \left( \frac{\sqrt{2}(x+1)}{\sqrt{7}} \right) + C$$