TRIGONOMETRIC INTEGRALS

SOME USEFUL IDENTITIES :

$$\cos^2 x + \sin^2 x = 1 \qquad (\cos^2 x = 1 - \sin^2 x, \text{ ETC.})$$

$$1 + \tan^2 x = \sec^2 x$$

$$\cot^2 x + 1 = \csc^2 x$$

$$\cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x)$$

TRIGONOMETRIC INTEGRALS :

NOTE : THERE ARE "REDUCTION FORMULAS" FOR $\int \sin^n x\,dx$, $\int \cos^n x\,dx$, $\int \tan^n x\,dx$ AND $\int \sec^n x\,dx$. NEED NOT BE MEMORIZED, BUT THESE OFTEN PROVIDE THE MOST EFFICIENT MEANS OF COMPUTING SUCH INTEGRALS.

IN ADDITION TO THE "BASIC TABLE OF INTEGRALS" THERE IS ONE MORE THAT COMES UP OFTEN ENOUGH (AND IS TRICKY ENOUGH) THAT YOU SHOULD REMEMBER IT :

$$\boxed{\int \sec x\,dx = \ln|\sec x + \tan x| + C}$$

HERE'S WHERE THIS COMES FROM :

$$\int \sec x \, dx = \int \sec x \, \frac{\sec x + \tan x}{\sec x + \tan x} \, dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx$$

$$u = \sec x + \tan x$$
$$du = (\sec x \tan x + \sec^2 x) \, dx$$

$$= \int \frac{1}{u} \, du = \ln |u| + C$$

$$= \ln |\sec x + \tan x| + C .$$

<u>MORE</u> <u>EXAMPLES</u> :

1. $\int \sin^4 x \cos^5 x \, dx$

<u>NOTE</u> : SINES AND COSINES WITH AT LEAST ONE ODD POWER.

$$\int \sin^4 x \cos^5 x \, dx = \int \sin^4 x \cos^4 x \cos x \, dx$$

$$= \int \sin^4 x \, (\cos^2 x)^2 \cos x \, dx$$

$$= \int \sin^4 x \, (1 - \sin^2 x)^2 \cos x \, dx$$

$$u = \sin x$$
$$du = \cos x \, dx$$

$$= \int u^4 (1 - u^2)^2 \, du$$

$$= \int u^4 (1 - 2u^2 + u^4) \, du$$

$$= \int (u^4 - 2u^6 + u^8)\,du$$

$$= \frac{1}{5}u^5 - \frac{2}{7}u^7 + \frac{1}{9}u^9 + C$$

$$= \frac{1}{5}\sin^5 x - \frac{2}{7}\sin^7 x + \frac{1}{9}\sin^9 x + C$$

2. $\int \cos^2 x\, dx$

COULD USE A REDUCTION FORMULA, OR

$$\int \cos^2 x\, dx = \int \frac{1}{2}(1 + \cos 2x)\,dx$$

$$= \frac{1}{2}\int 1\, dx + \frac{1}{2}\int \cos 2x\, dx$$

$u = 2x$
$du = 2dx$

$$= \frac{1}{2}x + \frac{1}{4}\int \cos u\, du$$

$$= \frac{1}{2}x + \frac{1}{4}\sin u + C$$

$$= \frac{1}{2}x + \frac{1}{4}\sin 2x + C .$$

3. $\int \sin^3 x\, dx$

COULD USE A REDUCTION FORMULA, OR

$$\int \sin^3 x\, dx = \int \sin^2 x \sin x\, dx = \int (1 - \cos^2 x)\sin x\, dx$$

$u = \cos x$
$du = -\sin x\, dx$

4. $\int \sin^2 x \cos^2 x \, dx$

NOTE : SINES AND COSINES, BOTH EVEN POWERS.

$$\int \sin^2 x \cos^2 x \, dx = \int \sin^2 x \,(1 - \sin^2 x)\, dx = \int \sin^2 x - \sin^4 x \, dx$$

NOW USE REDUCTION A FEW TIMES.

$$\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx$$

$n = 2$:
$$\int \sin^2 x \, dx = -\frac{1}{2} \sin x \cos x + \frac{1}{2} \int 1 \, dx$$
$$= -\frac{1}{2} \sin x \cos x + \frac{1}{2} x + C$$

$n = 4$:
$$\int \sin^4 x \, dx = -\frac{1}{4} \sin^3 x \cos x + \frac{3}{4} \int \sin^2 x \, dx$$
$$= -\frac{1}{4} \sin^3 x \cos x + \frac{3}{4} \left[-\frac{1}{2} \sin x \cos x + \frac{1}{2} x \right] + C$$
$$= -\frac{1}{4} \sin^3 x \cos x - \frac{3}{8} \sin x \cos x + \frac{3}{8} x + C$$

THUS,

$$\int \sin^2 x \cos^2 x \, dx = \left[-\frac{1}{2} \sin x \cos x + \frac{1}{2} x \right] - \left[-\frac{1}{4} \sin^3 x \cos x - \frac{3}{8} \sin x \cos x + \frac{3}{8} x \right] + C$$
$$= \frac{1}{4} \sin^3 x \cos x - \frac{1}{8} \sin x \cos x + \frac{1}{8} x + C$$

ALTERNATIVE SOLUTION :

$$\int \sin^2 x \cos^2 x \, dx = \int \left(\tfrac{1}{2}(1-\cos 2x)\right)\left(\tfrac{1}{2}(1+\cos 2x)\right) dx$$

$$= \frac{1}{4}\int (1-\cos^2 2x)\,dx = \frac{1}{4}\int \sin^2 2x \, dx = \frac{1}{4}\int \frac{1}{2}(1-\cos 4x)\,dx$$

$$= \frac{1}{8}\int (1-\cos 4x)\,dx = \frac{1}{8}\left(x - \frac{1}{4}\sin 4x\right) + C$$

$$= \frac{1}{8}x - \frac{1}{32}\sin 4x + C$$

5. $\int \tan^2 x \sec^4 x \, dx$

NOTE : TANGENTS AND SECANTS WITH AN EVEN POWER OF SECANT.

$$\int \tan^2 x \sec^2 x \sec^2 x \, dx = \int \tan^2 x \,(1+\tan^2 x)\sec^2 x \, dx =$$

$u = \tan x$
$du = \sec^2 x \, dx$

$$\int u^2(1+u^2)\,du = \int (u^2+u^4)\,du = \frac{1}{3}u^3 + \frac{1}{5}u^5 + C =$$

$$\frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + C$$

6. $\int \tan^3 x \sec^3 x \, dx$

NOTE : TANGENTS AND SECANTS, BOTH POWERS ODD.

$$\int \tan^2 x \sec^2 x (\sec x \tan x) dx = \int (\sec^2 x - 1) \sec^2 x (\sec x \tan x) dx =$$

$u = \sec x$

$du = \sec x \tan x$

$$\int (u^2 - 1) u^2 du = \int (u^4 - u^2) du = \frac{1}{5} u^5 - \frac{1}{3} u^3 + C =$$

$$\frac{1}{5} \sec^5 x - \frac{1}{3} \sec^3 x + C$$

7. $\int \sin 7x \cos 3x \, dx$

THIS ONE IS TRICKY. RECALL:

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

ADD

$$\sin(A - B) + \sin(A + B) = 2 \sin A \cos B$$

SO

$$\sin A \cos B = \frac{1}{2}(\sin(A - B) + \sin(A + B))$$

WHEN $A = 7x$ AND $B = 3x$ THIS BECOMES

$$\sin 7x \cos 3x = \frac{1}{2}(\sin 4x + \sin 10x)$$

$$\int \sin 7x \cos 3x \, dx = \frac{1}{2} \int (\sin 4x + \sin 10x) dx$$

$$= \frac{1}{2}\left[-\frac{1}{4}\cos 4x - \frac{1}{10}\cos 10x\right] + C$$

$$= -\frac{1}{8}\cos 4x - \frac{1}{20}\cos 10x + C$$

8. $\displaystyle\int \frac{\sec\sqrt{x}}{\sqrt{x}}\,dx = 2\int \sec u\,du = 2\ln|\sec u + \tan u| + C$

$= 2\ln|\sec\sqrt{x} + \tan\sqrt{x}| + C$

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

9. $\displaystyle\int \tan^2 x \sec x\,dx = \int (\sec^2 x - 1)\sec x\,dx =$

$\displaystyle\underbrace{\int \sec^3 x\,dx}_{\text{REDUCTION FORMULA}} - \int \sec x\,dx =$

$\displaystyle\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-2}x\tan x + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx$

$\displaystyle\frac{1}{2}\sec x\tan x + \frac{1}{2}\int \sec x\,dx - \int \sec x\,dx =$

$\displaystyle\frac{1}{2}\sec x\tan x - \frac{1}{2}\int \sec x\,dx =$

$\displaystyle\frac{1}{2}\sec x\tan x - \frac{1}{2}\ln|\sec x + \tan x| + C$