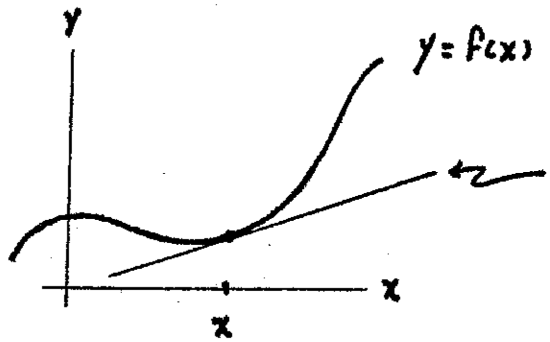
THE DERIVATIVE



SLOPE OF TANGENT LINE TO THE GRAPH OF f AT $(x, f(x))$

$=$ INSTANTANEOUS RATE OF CHANGE OF f AT x

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ (PROVIDED THE LIMIT EXISTS)

NEW STUFF :

$=$ THE **DERIVATIVE OF** f **AT** x

$= f'(x)$

$= y'$

$= \dfrac{dy}{dx}$

$= \dfrac{d}{dx}(f(x))$

EXAMPLE : LET $f(x) = \sqrt{x}$ FOR $x \geq 0$. THEN

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \, \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$ (PROVIDED $x \neq 0$)

THUS,

$$f(x) = \sqrt{x} \text{ FOR } x \geq 0 \implies f'(x) = \frac{1}{2\sqrt{x}} \text{ FOR } x > 0$$

THE PROCESS OF FINDING DERIVATIVES OF FUNCTIONS IS CALLED

<u>DIFFERENTIATION</u>

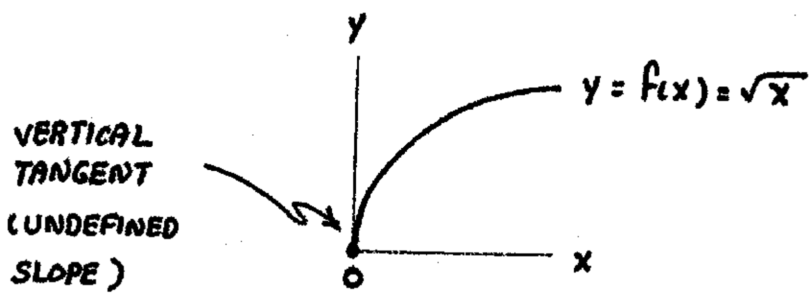
IF A FUNCTION HAS A DERIVATIVE AT A POINT IT IS SAID TO BE

<u>DIFFERENTIABLE</u>

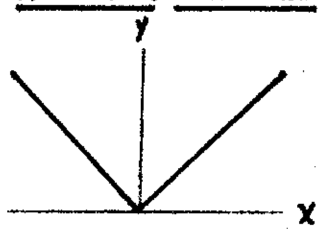
AT THAT POINT

E.G., $f(x) = \sqrt{x}$ IS DIFFERENTIABLE AT EVERY POINT IN ITS DOMAIN <u>EXCEPT</u> $x = 0$.

GEOMETRICAL REASON :



<u>ANOTHER</u> <u>EXAMPLE</u> : $f(x) = |x|$ IS NOT DIFFERENTIABLE AT $x = 0$



$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{|0+h| - |0|}{h},$$

$$= \lim_{h \to 0} \frac{|h|}{h}$$

WHICH DOES NOT EXIST BECAUSE

$$\frac{|h|}{h} = \begin{cases} 1, & h > 0 \\ -1, & h < 0 \end{cases}$$

ANOTHER WAY FOR DIFFERENTIABILITY TO FAIL:

<u>THEOREM</u>: IF $f(x)$ IS DIFFERENTIABLE AT x_0, THEN $f(x)$ IS CONTINUOUS AT x_0

(SO, IF $f(x)$ IS <u>NOT</u> CONTINUOUS AT x_0, THEN $f(x)$ IS <u>NOT</u> DIFFERENTIABLE AT x_0)

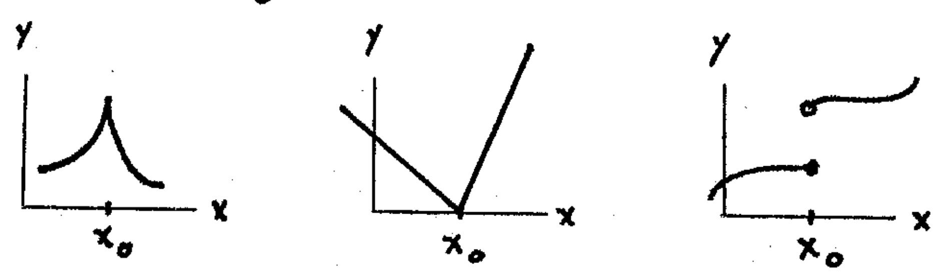
REASON: IF $\lim\limits_{x \to x_0} \dfrac{f(x)-f(x_0)}{x-x_0} = f'(x_0)$, THEN

$$\begin{aligned}
\lim_{x \to x_0} (f(x)-f(x_0)) &= \lim_{x \to x_0} \left(\frac{f(x)-f(x_0)}{x-x_0}\right)(x-x_0) \\
&= \left(\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}\right)\left(\lim_{x \to x_0} (x-x_0)\right) \\
&= f'(x_0) \cdot 0 \\
&= 0
\end{aligned}$$

SO

$$\lim_{x \to x_0} f(x) = f(x_0).$$

HERE, THEN, ARE THE WAYS IN WHICH $f(x)$ CAN FAIL TO BE DIFFERENTIABLE AT x_0:



MORE EXAMPLES :

1. COMPUTE $\frac{dy}{dx}$ IF $y = f(x) = x^2 + 2x - 1$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{((x+h)^2 + 2(x+h) - 1) - (x^2 + 2x - 1)}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + 2x + 2h - 1 - x^2 - 2x + 1}{h}$$

$$= \lim_{h \to 0} \frac{h(2x + h + 2)}{h} = \lim_{h \to 0} (2x + h + 2) = 2x + 2$$

2. FIND THE EQUATION OF THE TANGENT LINE TO THE GRAPH OF

$$y = f(x) = x^2 + 2x - 1$$

AT THE POINT $(1, 2)$.

POINT - SLOPE FORM : $y - y_0 = m(x - x_0)$

$y_0 \to 2$, $x_0 \to 1$

$m \to$ SLOPE OF TANGENT LINE AT $(1,2)$

$= f'(1)$

$= 2(1) + 2$ (FROM EXAMPLE 1)

$= 4$

$$y - 2 = 4(x - 1)$$

$$y = 4x - 2$$

3. FIND THE POINT ON THE GRAPH OF $y = f(x) = x^2 + 2x - 1$ WHERE THE TANGENT LINE IS HORIZONTAL.

TANGENT LINE HORIZONTAL $\iff$ SLOPE OF TANGENT LINE ZERO

$\iff$ DERIVATIVE ZERO

$\iff$ $f'(x) = 0$

$2x + 2 = 0$ (FROM EXAMPLE 1)

$x = -1$

$y = f(-1) = (-1)^2 + 2(-1) - 1$

$= -2$

ANSWER : $(-1, -2)$

4. COMPUTE $g'(t)$ IF $g(t) = \frac{1}{t}$

$$g'(t) = \lim_{h \to 0} \frac{g(t+h) - g(t)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{t+h} - \frac{1}{t}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{t+h} - \frac{1}{t}}{h} \; \frac{t(t+h)}{t(t+h)}$$

$$= \lim_{h \to 0} \frac{t - (t+h)}{ht(t+h)} = \lim_{h \to 0} \frac{-1}{t(t+h)}$$

$$= -\frac{1}{t^2}$$