TECHNIQUES FOR COMPUTING LIMITS

EXAMPLES :

1. $\lim_{x \to 3} x^2 = 3^2 = 9$

2. $\lim_{x \to -1} (x^2 + 2x) = (-1)^2 + 2(-1) = -1$

POLYNOMIALS : $P(x) = a_n x^n + \cdots + a_1 x + a_0$

$$\lim_{x \to a} P(x) = P(a)$$
$$= a_n a^n + \cdots + a_1 a + a_0$$

3. $\lim_{x \to 3} \dfrac{x^2 - 2x}{x+1} \quad \begin{matrix} \longrightarrow 3^2 - 2(3) = 3 \\ \longrightarrow 3 + 1 = 4 \end{matrix} \quad = \dfrac{3}{4}$

4. $\lim_{x \to 1} \dfrac{x^2 - 4x + 3}{x^2 - 2x - 3} \quad \begin{matrix} \longrightarrow 1^2 - 4(1) + 3 = 0 \\ \longrightarrow 1^2 - 2(1) - 3 = -4 \end{matrix} \quad = \dfrac{0}{-4} = 0$

5. $\lim_{x \to 1} \dfrac{x^2 - 2x - 3}{x^2 - 4x + 3} \quad \begin{matrix} \longrightarrow -4 \\ \longrightarrow 0 \end{matrix}$ SO THE LIMIT DOES NOT EXIST

RATIONAL FUNCTIONS: $P(x)$ AND $Q(x)$ POLYNOMIALS.

$$\lim_{x\to a} \frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)} \quad \text{UNLESS } Q(a)=0$$

IF $Q(a) = 0$, BUT $P(a) \neq 0$, THE LIMIT DOES NOT EXIST.

IF $Q(a) = 0$ AND $P(a) = 0$ THE LIMIT IS SAID TO BE INDETERMINATE OF TYPE

$$\frac{0}{0}$$

AND MORE WORK IS REQUIRED TO DETERMINE THE LIMIT

E.G., ALL OF THE FOLLOWING ARE INDETERMINATE

$$\lim_{x\to 0} \frac{x^3}{x}, \quad \lim_{x\to 0} \frac{x}{x^3}, \quad \lim_{x\to 0} \frac{x^3}{x^3}, \quad \lim_{x\to 0} \frac{5x^3}{x^3}$$

6. $\displaystyle\lim_{x\to 1} \frac{2x+1}{x^2-1}$ $\begin{array}{l} \to 2(1)+1 = 3 \\ \to 1^2 - 1 = 0 \end{array}$ SO THE LIMIT DNE.

NOTE THAT IN THIS CASE WE CAN DO BETTER:

$$\lim_{x\to 1^+} \frac{2x+1}{x^2-1} = \infty$$

$$\lim_{x\to 1^-} \frac{2x+1}{x^2-1} = -\infty$$

7. $\displaystyle \lim_{x \to 3} \frac{x^2-2x-3}{x^2-4x+3} \begin{array}{l} \to 0 \\ \to 0 \end{array}$

THIS TIME THE "EXTRA WORK" REQUIRED TO DETERMINE THE LIMIT IS PRETTY OBVIOUS :

FACTOR AND CANCEL

$$\lim_{x \to 3} \frac{x^2-2x-3}{x^2-4x+3} = \lim_{x \to 3} \frac{(x-3)(x+1)}{(x-3)(x-1)}$$

$$= \lim_{x \to 3} \frac{x+1}{x-1}$$

NOTE : CANCELLING THE $x-3$ MEANS DIVIDING NUMERATOR AND DENOMINATOR BY $x-3$. THIS IS LEGITIMATE BECAUSE, IN $\lim\limits_{x \to 3}$, $x \neq 3$.

$$= \frac{3+1}{3-1} = 2$$

8. $\displaystyle \lim_{t \to -2} \frac{t^3+8}{t+2} \begin{array}{l} \to 0 \\ \to 0 \end{array}$

$$= \lim_{t \to -2} \frac{(t+2)(t^2-2t+4)}{t+2}$$

RECALL :

$A^3+B^3 = (A+B)(A^2-AB+B^2)$

$A^3-B^3 = (A-B)(A^2+AB+B^2)$

$$= \lim_{t \to -2} (t^2-2t+4)$$

$$= (-2)^2-2(-2)+4 = 12$$

9. $$\lim_{h\to 0} \frac{(2+h)^3-8}{h} \quad \begin{matrix}\to 0\\ \to 0\end{matrix}$$

THIS TIME IT IS PROBABLY EASIEST TO

EXPAND

$$\lim_{h\to 0} \frac{(2+h)^3-8}{h} = \lim_{h\to 0} \frac{(8+12h+6h^2+h^3)-8}{h}$$

$$= \lim_{h\to 0} \frac{12h+6h^2+h^3}{h} = \lim_{h\to 0} \frac{h(12+6h+h^2)}{h}$$

$$= \lim_{h\to 0} (12+6h+h^2) = 12+6\cdot 0+0^2 = 12$$

10. $$\lim_{x\to 0} \frac{\sqrt{1-x}-1}{x} \quad \begin{matrix}\to 0\\ \to 0\end{matrix}$$

THE STANDARD THING TO DO WITH A SQUARE ROOT IN A SUM OR DIFFERENCE IS

RATIONALIZE

$$\lim_{x\to 0} \frac{\sqrt{1-x}-1}{x} = \lim_{x\to 0} \frac{\sqrt{1-x}-1}{x} \frac{\sqrt{1-x}+1}{\sqrt{1-x}+1} = \lim_{x\to 0} \frac{(\sqrt{1-x})^2-1^2}{x(\sqrt{1-x}+1)}$$

$$= \lim_{x\to 0} \frac{1-x-1}{x(\sqrt{1-x}+1)} = \lim_{x\to 0} \frac{-x}{x(\sqrt{1-x}+1)} = \lim_{x\to 0} \frac{-1}{\sqrt{1-x}+1}$$

$$= \frac{-1}{\sqrt{1-0}+1} = -\frac{1}{2}$$

11. $$\lim_{t\to 0} \frac{\frac{1}{2+t^2} - \frac{1}{2}}{t} \quad \begin{matrix} \to 0 \\ \to 0 \end{matrix}$$

<u>SIMPLIFY THE COMPLEX FRACTION</u>

$$\lim_{t\to 0} \frac{\frac{1}{2+t^2} - \frac{1}{2}}{t} = \lim_{t\to 0} \frac{\frac{1}{2+t^2} - \frac{1}{2}}{t} \cdot \frac{2(2+t^2)}{2(2+t^2)}$$

$$= \lim_{t\to 0} \frac{2-(2+t^2)}{2t(2+t^2)} = \lim_{t\to 0} \frac{-t^2}{2t\,2+t^2)}$$

$$= \lim_{t\to 0} \frac{-t}{2(2+t^2)} \quad \begin{matrix} \to 0 \\ \to 4 \end{matrix} \quad = \frac{0}{4} = 0$$

12. $$f(x) = \begin{cases} \frac{1}{x+2} & , x < -2 \\ x^2-5 & , -2 < x \le 3 \\ \sqrt{x+13} & , x > 3 \end{cases}$$

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} \frac{1}{x+2} = -\infty$$

$$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} (x^2-5) = (-2)^2-5 = -1$$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} (x^2-5) = 0^2-5 = -5$$

$$\left.\begin{matrix} \lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (x^2-5) = 3^2-5 = 4 \\ \lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \sqrt{x+13} = \sqrt{3+13} = 4 \end{matrix}\right\} \quad \text{THUS, } \lim_{x\to 3} f(x) = 4$$