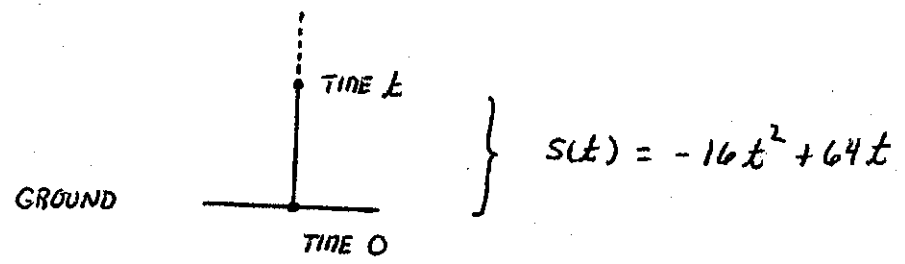
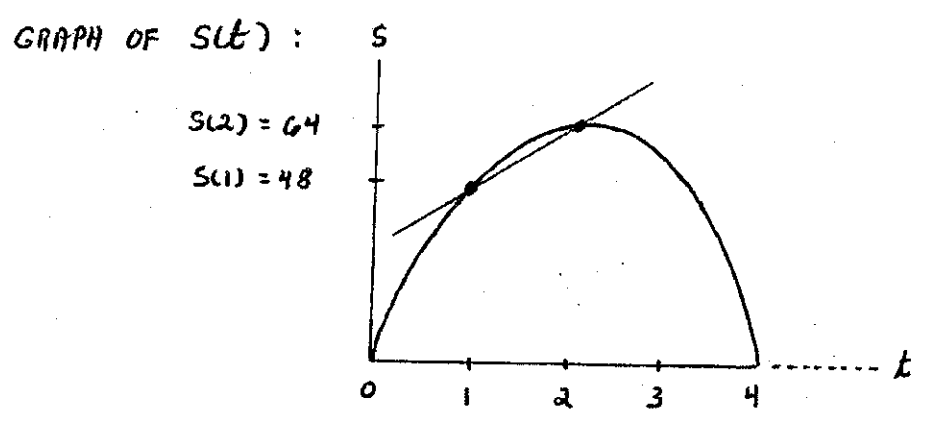


TANGENT LINES AND RATES OF CHANGE

HERE'S A FACT FROM PHYSICS : A BALL IS THROWN VERTICALLY UPWARD FROM THE GROUND AT AN INITIAL VELOCITY OF 64 FT/SEC. THEN t SEC LATER ITS HEIGHT S IN FEET IS GIVEN BY



E.G., $S(1) = -16 \cdot 1^2 + 64 \cdot 1 = 48$
 $S(2) = -16 \cdot 2^2 + 64 \cdot 2 = 64$
 $S(3) = -16 \cdot 3^2 + 64 \cdot 3 = 48$
 $S(4) = -16 \cdot 4^2 + 64 \cdot 4 = 0$



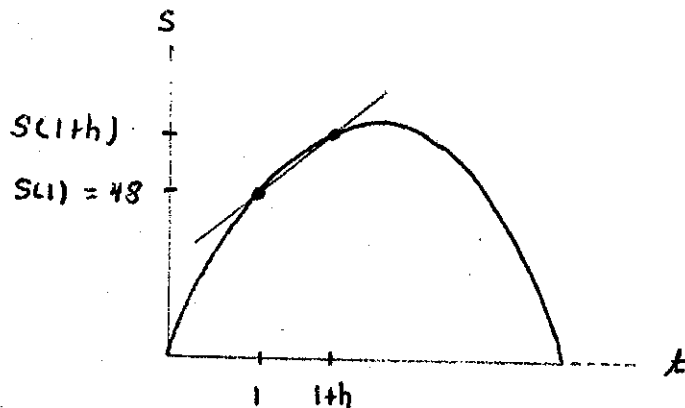
NOTE : AVERAGE VELOCITY OF THE BALL OVER THE TIME INTERVAL FROM $t=1$ TO $t=2$ IS

$$\frac{\text{DISTANCE TRAVELED}}{\text{LENGTH OF TIME INTERVAL}} = \frac{S(2) - S(1)}{2 - 1} = \frac{64 - 48}{1} = 16 \text{ FT/SEC}$$

WHICH IS ALSO THE SLOPE OF THE STRAIGHT LINE JOINING THE POINTS $(1, 48)$ AND $(2, 64)$ ON THE GRAPH OF $S(t)$.

SIMILARLY, THE AVERAGE VELOCITY OF THE BALL BETWEEN $t=2$ AND $t=3$ IS -16 FT/SEC AND THIS IS THE SLOPE OF THE LINE JOINING $(2, 64)$ AND $(3, 48)$.

NEXT, LET'S COMPUTE THE AVERAGE VELOCITY OF THE BALL OVER SOME "SMALL" TIME INTERVAL FROM $t=1$ TO $t=1+h$.

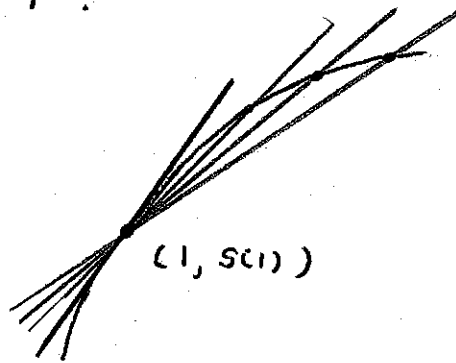


$$\begin{aligned} \frac{s(1+h) - s(1)}{h} &= \frac{-16(1+h)^2 + 64(1+h) - 48}{h} \\ &= \frac{-16(1+2h+h^2) + 64 + 64h - 48}{h} \\ &= \frac{-16 - 32h - 16h^2 + 64 + 64h - 48}{h} \\ &= \frac{-16h^2 + 32h}{h} = -16h + 32 \end{aligned}$$

= SLOPE OF THE LINE JOINING THE POINTS $(1, s(1))$ AND $(1+h, s(1+h))$ ON THE GRAPH OF $s(t)$.

THIS IS TRUE FOR ANY (NONZERO) h . IF h IS VERY SMALL, THE BALL HAS VERY LITTLE TIME TO SLOW DOWN OR SPEED UP AND THIS AVERAGE VELOCITY IS VERY CLOSE TO THE BALL'S VELOCITY " AT $t = 1$ " (" SPEEDOMETER READING " , " INSTANTANEOUS VELOCITY ")

AS $h \rightarrow 0$ THE AVERAGE VELOCITY APPROACHES THE INSTANTANEOUS VELOCITY AT $t = 1$:



INSTANTANEOUS VELOCITY AT $t = 1$

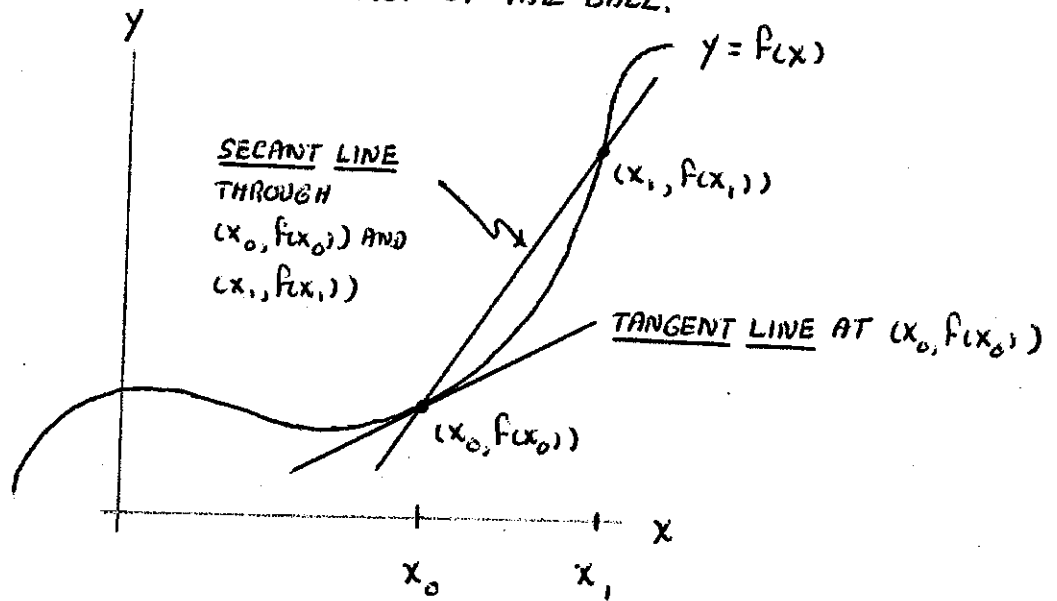
$$= \lim_{h \rightarrow 0} \frac{s(1+h) - s(1)}{h}$$

$$= \lim_{h \rightarrow 0} (-16h + 32)$$

$$= 32$$

$$= \underline{\text{SLOPE OF THE TANGENT LINE}} (1, s(1))$$

NOW WE WILL DO FOR AN ARBITRARY FUNCTION $f(x)$ WHAT WE JUST DID FOR THE POSITION FUNCTION OF THE BALL.



AVERAGE RATE OF CHANGE OF $f(x)$ OVER THE INTERVAL $[x_0, x_1]$

$$= \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

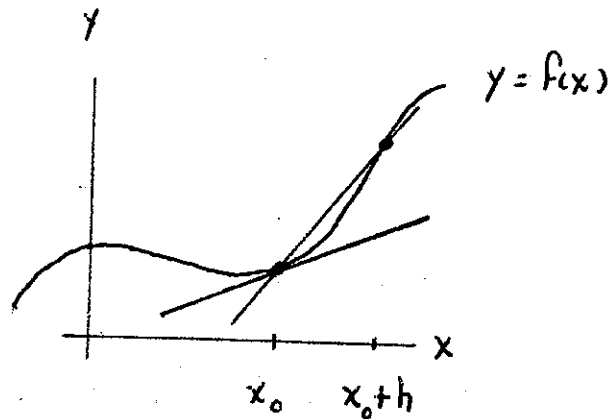
$$= \text{SLOPE OF SECANT LINE THROUGH } (x_0, f(x_0)) \text{ AND } (x_1, f(x_1))$$

INSTANTANEOUS RATE OF CHANGE OF $f(x)$ AT x_0

$$= \lim_{x_1 \rightarrow x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} \quad (\text{PROVIDED THE LIMIT EXISTS})$$

$$= \text{SLOPE OF TANGENT LINE AT } (x_0, f(x_0))$$

SOMETIMES IT IS MORE CONVENIENT TO USE THE FOLLOWING EQUIVALENT FORMULA :



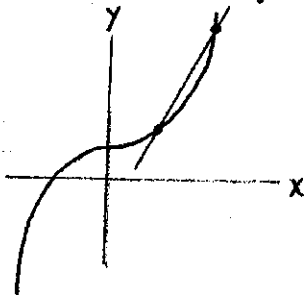
INSTANTANEOUS RATE OF CHANGE OF $f(x)$ AT x_0

$$= \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} \quad (\text{PROVIDED THE LIMIT EXISTS})$$

$$= \text{SLOPE OF TANGENT LINE AT } (x_0, f(x_0))$$

EXAMPLES :

1. FIND THE AVERAGE RATE OF CHANGE OF $f(x) = x^3 + 1$ OVER THE INTERVAL $[1, 2]$



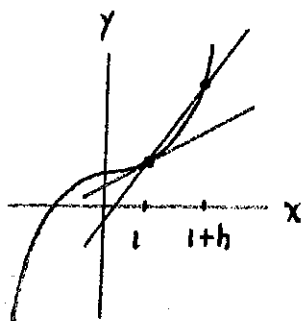
$$[x_0, x_1] = [1, 2]$$

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{(x_1^3 + 1) - (x_0^3 + 1)}{x_1 - x_0}$$

$$= \frac{(2^3 + 1) - (1^3 + 1)}{2 - 1}$$

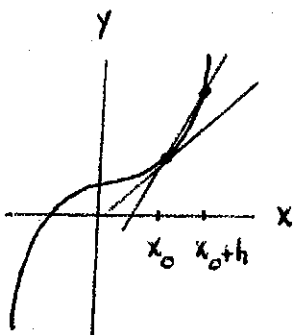
$$= 7$$

2. FIND THE INSTANTANEOUS RATE OF CHANGE OF $f(x) = x^3 + 1$ AT $x_0 = 1$.



$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h} &= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(1+h)^3 + 1 - (1^3 + 1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(1 + 3h + 3h^2 + h^3 + 1) - 2}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(3 + 3h + h^2)}{h} = \lim_{h \rightarrow 0} (3 + 3h + h^2) = 3 \end{aligned}$$

3. FIND THE INSTANTANEOUS RATE OF CHANGE OF $f(x) = x^3 + 1$ AT x_0 .



$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h} &= \lim_{h \rightarrow 0} \frac{(x_0+h)^3 + 1 - (x_0^3 + 1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{x_0^3 + 3x_0^2h + 3x_0h^2 + h^3 + 1 - x_0^3 - 1}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(3x_0^2 + 3x_0h + h^2)}{h} \\ &= \lim_{h \rightarrow 0} (3x_0^2 + 3x_0h + h^2) \\ &= 3x_0^2 \end{aligned}$$