SURFACES IN SPACE

GRAPHS IN SPACE OF EQUATIONS IN x, y AND z, E.G.,

$x^2+y^2+z^2=1$ (KNOWN TO BE A SPHERE)

$x^2+y^2=1$ (KNOWN TO BE A CYLINDER)

$2x+3y=5$ (KNOWN TO BE A PLANE)

$z=x^2+y^2$

$x^2+9y^2+4z^2=36$

$\vdots$

PLOTTING POINTS TO OBTAIN THE GRAPH IS GENERALLY USELESS.

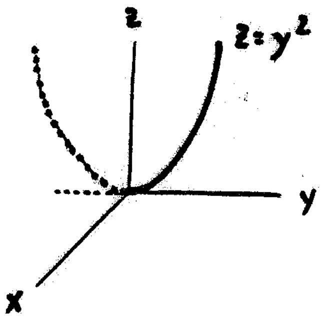
PROCEDURE : INTERSECT THE SURFACE WITH COORDINATE PLANES (BY SETTING $x, y, z = 0$) AND VARIOUS OTHER PLANES OF CONSTANT x, y, AND z TO GET CURVES (TRACES OF THE SURFACE) INDICATING THE SHAPE OF THE SURFACE. SOMETIMES BEST TO INITIALLY RESTRICT ATTENTION TO THE FIRST OCTANT AND LATER DRAW THE GLOBAL PICTURE.

EXAMPLES :

1. $z=x^2+y^2$

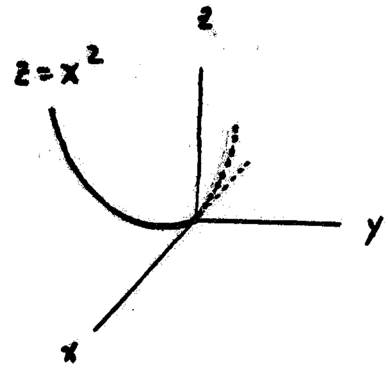
INTERSECTION WITH YZ - PLANE : $x = 0$

$$z = 0^2 + y^2$$
$$z = y^2$$



INTERSECTION WITH XZ - PLANE : $y = 0$

$$z = x^2 + 0^2$$
$$z = x^2$$



INTERSECTION WITH XY - PLANE : $z = 0$

$$0 = x^2 + y^2$$
$$(x, y) = (0, 0)$$

A POINT (THE ORIGIN)

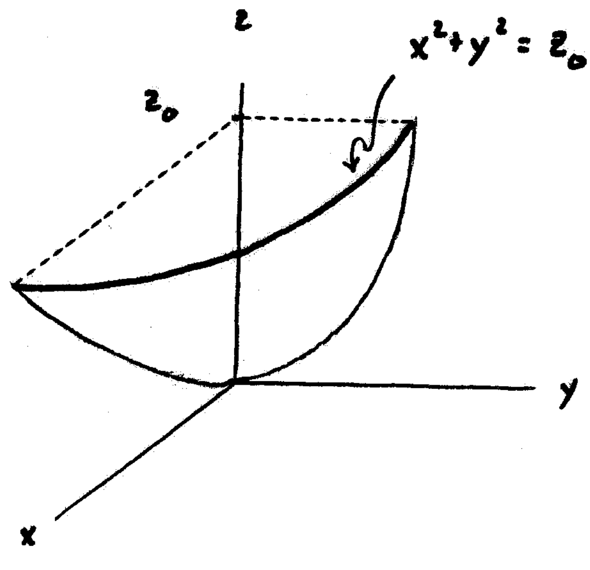
INTERSECTION WITH THE PLANE AT "HEIGHT" z_0 : $z = z_0$

$$x^2 + y^2 = z_0$$

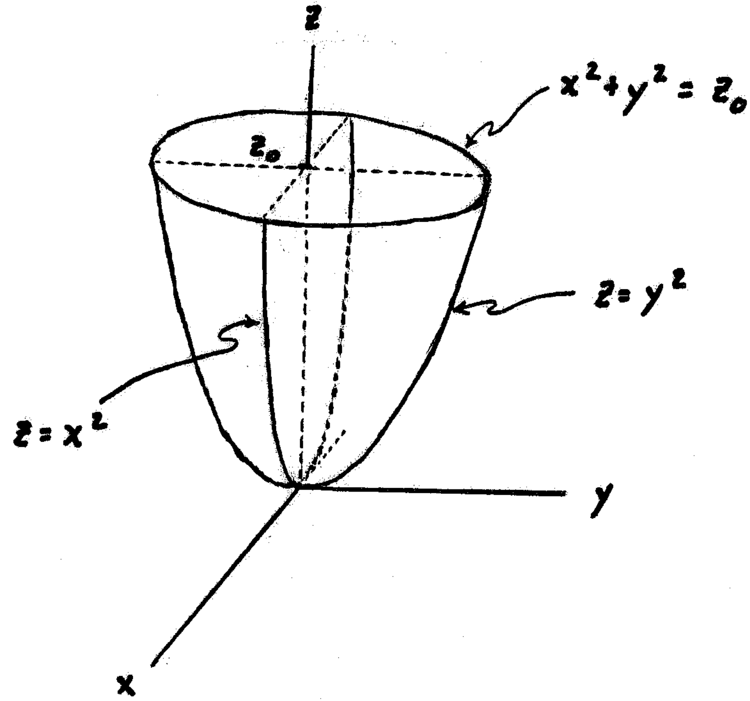
EMPTY IF $z_0 < 0$

A POINT IF $z_0 = 0$

A CIRCLE IF $z_0 > 0$



GLOBALLY, THE GRAPH IS A STACK OF CIRCLES CLIMBING UP PARABOLIC SIDES : A "BOWL" (TECHNICALLY, A <u>CIRCULAR</u> <u>PARABOLOID</u>).

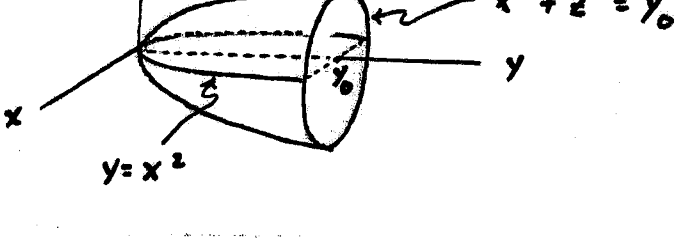


2. $z = 2x^2 + 3y^2$

GRAPH IS SIMILAR TO #1 EXCEPT THAT THE CROSS-SECTIONS AT HEIGHT z_0 ARE ELLIPSES $2x^2+3y^2 = z_0$. <u>ELLIPTICAL</u> <u>PARABOLOID</u>

3. $y = x^2 + z^2$

A CIRCULAR PARABOLOID, BUT ALONG THE y-AXIS (CROSS-SECTIONS AT $y = y_0$ ARE CIRCLES $x^2+z^2 = y_0$).



4. $z = \sqrt{x^2+y^2}$

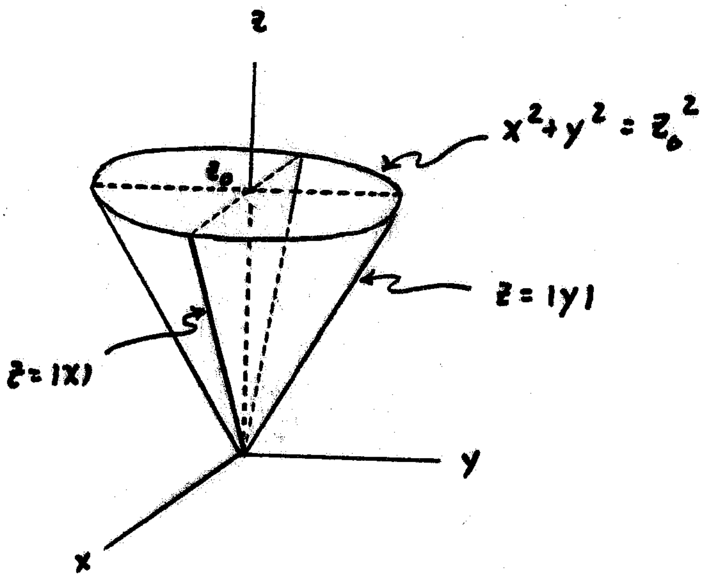
$x = 0 : z = \sqrt{y^2}$
$z = |y|$

$y = 0 : z = \sqrt{x^2}$
$z = |x|$

$z = z_0 : \sqrt{x^2+y^2} = z_0$
$x^2+y^2 = z_0^2$

A STACK OF CIRCLES CLIMBING UP STRAIGHT LINE SIDES :

<u>CIRCULAR</u> <u>CONE</u>



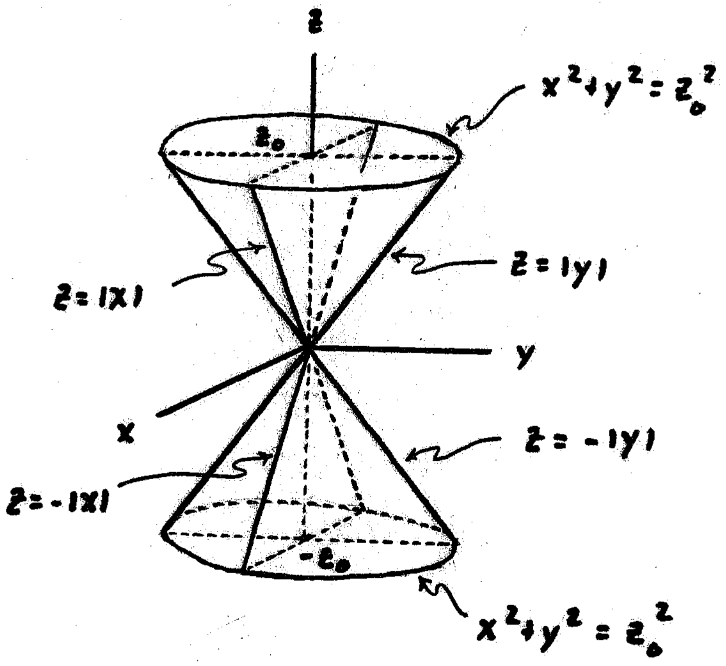
5. $x = \sqrt{2y^2+z^2}$ <u>ELLIPTICAL</u> <u>CONE</u> (ALONG X-AXIS)

6. $z^2 = x^2+y^2$ (OR $x^2+y^2-z^2 = 0$)

TAKE SQUARE ROOTS TO GET

$$z = \sqrt{x^2+y^2} \quad \text{OR} \quad z = -\sqrt{x^2+y^2}$$

SO THE GRAPH IS A <u>DOUBLE</u> <u>CIRCULAR</u> <u>CONE</u> :



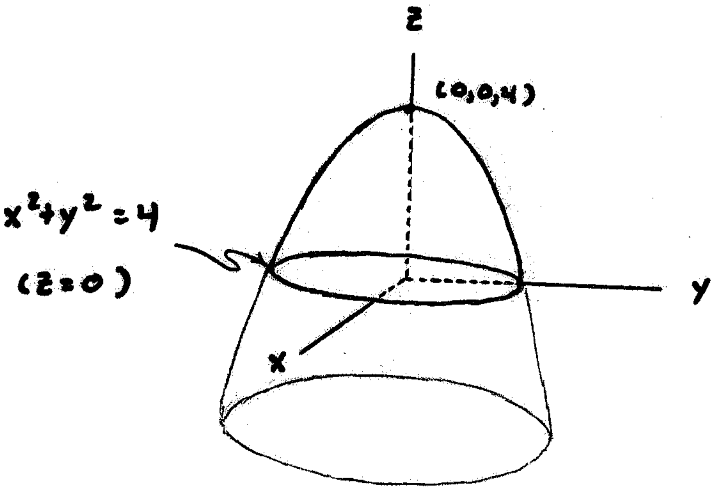
7. $-x^2 + 2y^2 + z^2 = 0$

<u>DOUBLE ELLIPTICAL CONE</u> (ALONG THE X-AXIS)

8. $z = 4 - x^2 - y^2$

REWRITE AS

$$z = \underbrace{\underbrace{-\underbrace{(x^2+y^2)}_{\text{BOWL}}}_{\text{INVERTED BOWL}} + 4}_{\text{INVERTED BOWL LIFTED 4 UNITS UP ON THE Z-AXIS}}$$

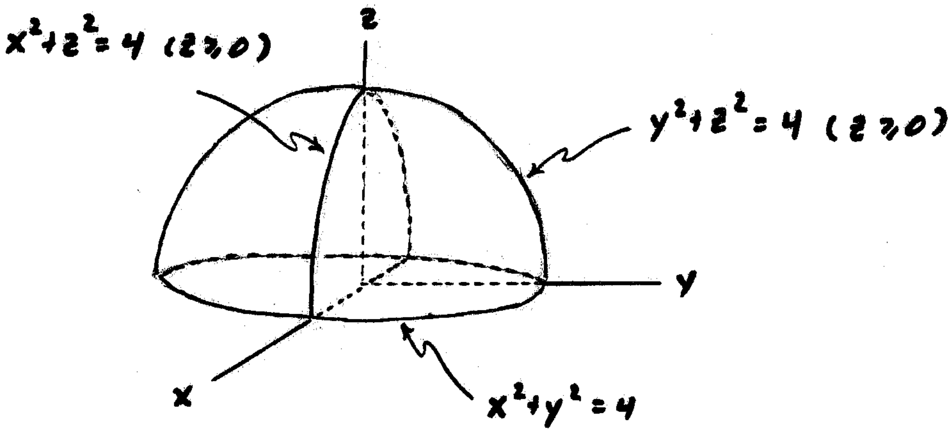


9. $z = \sqrt{4-x^2-y^2}$

SQUARE BOTH SIDES : $z^2 = 4-x^2-y^2$

$$x^2+y^2+z^2=4$$

SPHERE OF RADIUS 2 ABOUT THE ORIGIN, BUT ONLY THE TOP HALF ($z \geq 0$)



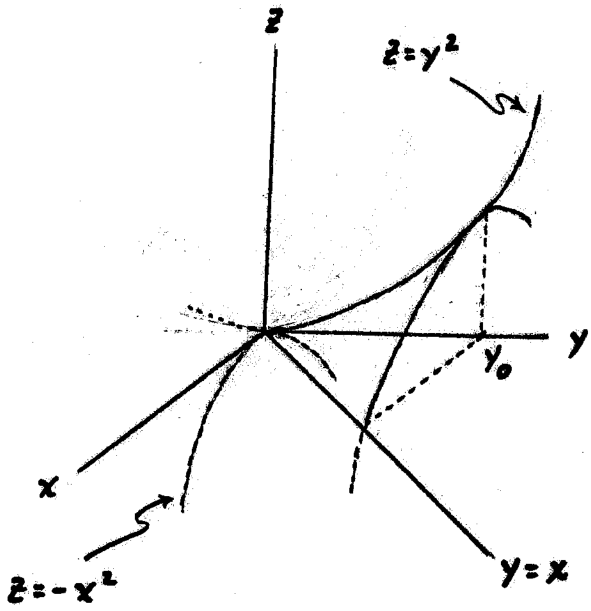
10. $z = y^2 - x^2$

$x = 0 \;:\; z = y^2$

$y = 0 \;:\; z = -x^2$

$z = 0 \;:\; y^2 - x^2 = 0$

$\qquad\qquad y = \pm x$



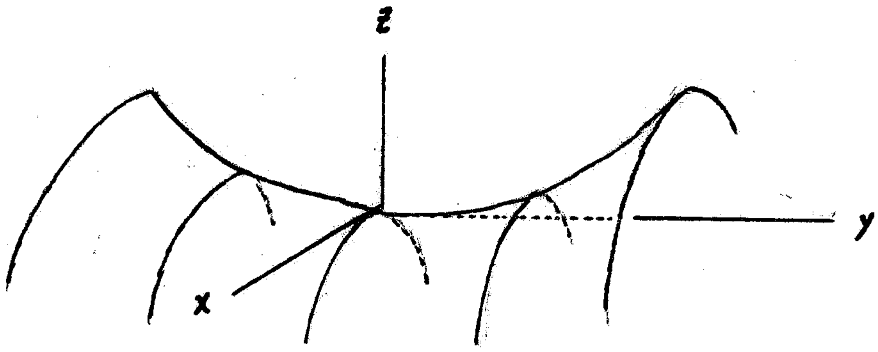
NOW INTERSECT WITH $y = y_0$ (TO SEE HOW $z = y^2$ AND $y = x$ ARE "CONNECTED" ON THE SURFACE) :

$$z = y_0^2 - x^2$$

$$z = -x^2 + y_0^2$$

(SAME PARABOLA AS $z = -x^2$, BUT LIFTED UP y_0^2 UNITS)

PICTURE THE SURFACE AS FOLLOWS : THE PARABOLA $z = y^2$ HAS, AT EACH POINT, A COPY OF THE PARABOLA $z = -x^2$ HANGING BENEATH IT AND THESE HANGING PARABOLAS INTERSECT THE XY-PLANE ALONG THE LINES $y = x$ AND $y = -x$.



IT'S A "SADDLE". TECHNICALLY, A <u>HYPERBOLIC</u> <u>PARABOLOID</u>

(NOTICE THAT THE INTERSECTION WITH A PLANE OF CONSTANT HEIGHT $z = z_0$ IS A HYPERBOLA $y^2 - x^2 = z_0$).

WE WILL DO THREE MORE EXAMPLES. THE EQUATIONS ARE SIMILAR, BUT THE GRAPHS ARE QUITE DIFFERENT.

11. $x^2 + 9y^2 + 4z^2 = 36$ $\quad \left(\frac{x^2}{6^2} + \frac{y^2}{2^2} + \frac{z^2}{3^2} = 1 \right)$

12. $x^2 + 9y^2 - 4z^2 = 36$ $\quad \left(\frac{x^2}{6^2} + \frac{y^2}{2^2} - \frac{z^2}{3^2} = 1 \right)$

13. $-x^2 - 9y^2 + 4z^2 = 36$ $\quad \left(-\frac{x^2}{6^2} - \frac{y^2}{2^2} + \frac{z^2}{3^2} = 1 \right)$

11. $x^2 + 9y^2 + 4z^2 = 36$

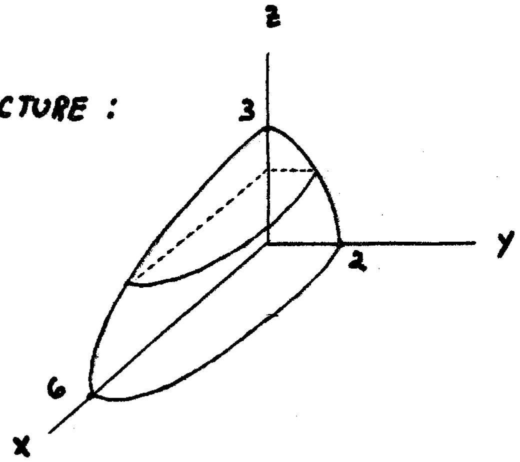
$x = 0$: $9y^2 + 4z^2 = 36$ ELLIPSE WITH INTERCEPTS AT $y = \pm 2$ AND $z = \pm 3$

$y = 0$: $x^2 + 4z^2 = 36$ ELLIPSE WITH INTERCEPTS AT $x = \pm 6$ AND $z = \pm 3$

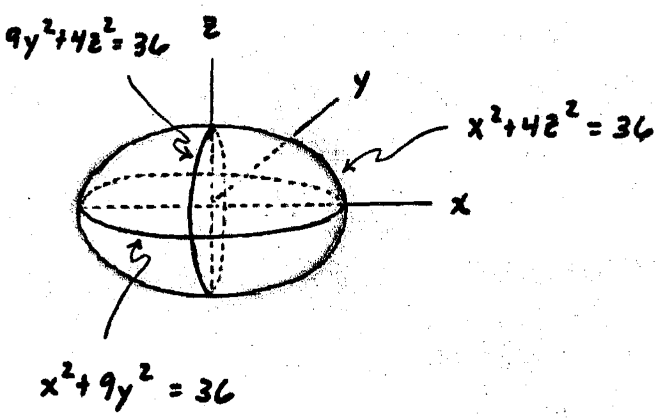
$z = 0$: $x^2 + 9y^2 = 36$ ELLIPSE WITH INTERCEPTS AT $x = \pm 6$ AND $y = \pm 2$

$z = z_0$: $x^2 + 9y^2 = 36 - 4z_0^2$ THIS IS AN ELLPSE IF $-3 < z_0 < 3$, A POINT IF $z_0 = -3$ OR $z_0 = 3$, AND EMPTY IF $z_0 < -3$ OR $z_0 > 3$

1ST OCTANT PICTURE :



GLOBALLY, IT'S AN EGG (ELLIPSOID) :



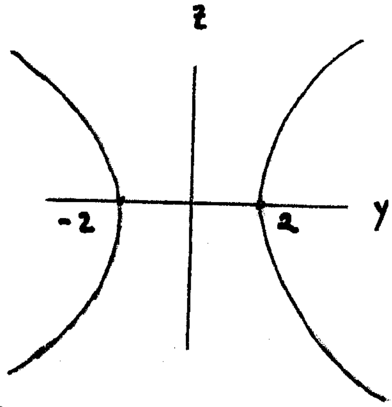
IN GENERAL,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

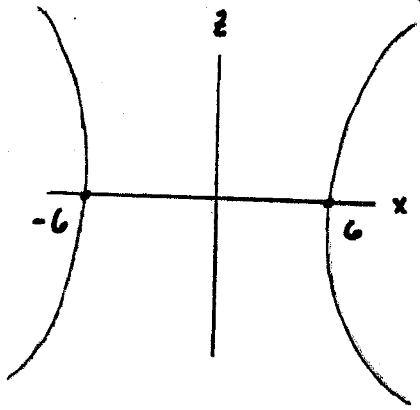
IS AN ELLIPSOID WITH INTERCEPTS AT $x = \pm a$, $y = \pm b$, $z = \pm c$.

12. $x^2 + 9y^2 - 4z^2 = 36$

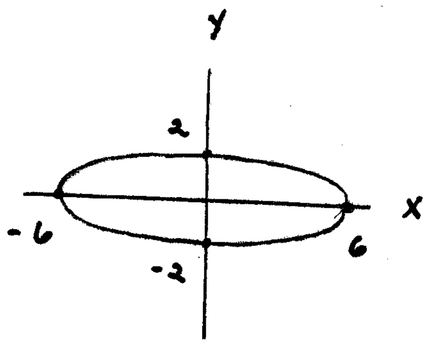
$x = 0$: $9y^2 - 4z^2 = 36$: HYPERBOLA



$y = 0$: $x^2 - 4z^2 = 36$: HYPERBOLA

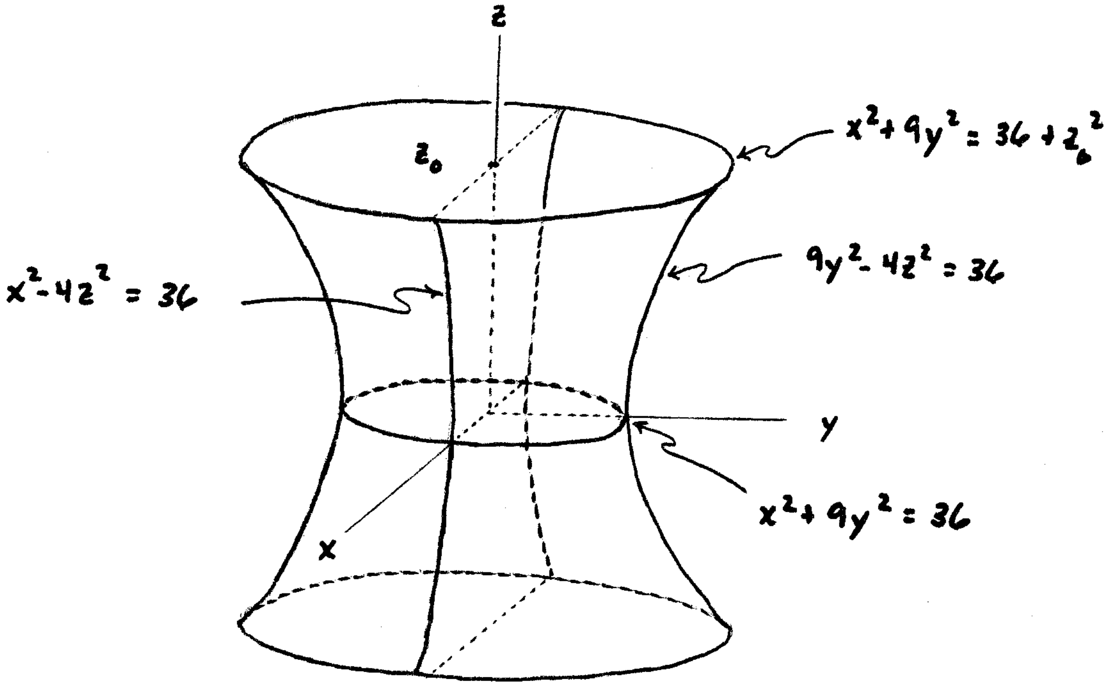


$z = 0$: $x^2 + 9y^2 = 36$: ELLIPSE



$z = z_0$: $x^2 + 9y^2 = 36 + z_0^2$

ALL ELLIPSES. SMALLEST AT $z_0 = 0$. GROWING AS $z_0 \to \infty$ AND $z_0 \to -\infty$.



$$x^2+9y^2-4z^2=36$$

<u>**HYPERBOLOID OF ONE SHEET**</u>

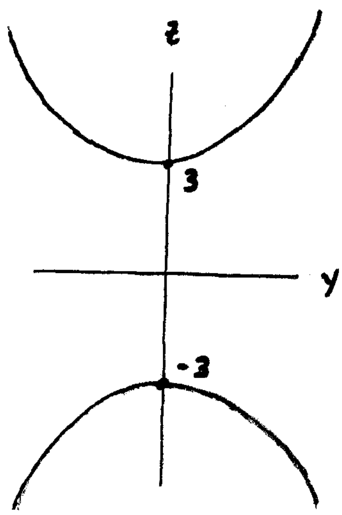
SIMILARLY FOR ANY

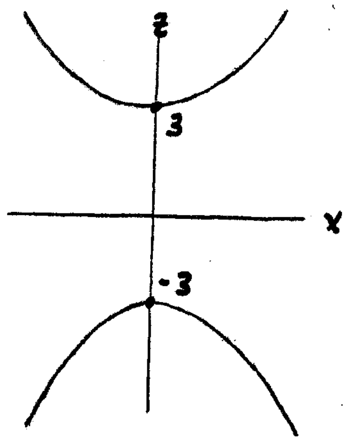
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

$$\left(\text{OR}\quad \frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad \text{OR} \quad -\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1\right)$$

13. $-x^2-9y^2+4z^2=36$

$x=0$: $-9y^2+4z^2=36$ HYPERBOLA





$y = 0: \quad -x^2 + 4z^2 = 36$ HYPERBOLA

$z = 0: \quad -x^2 - 9y^2 = 36$

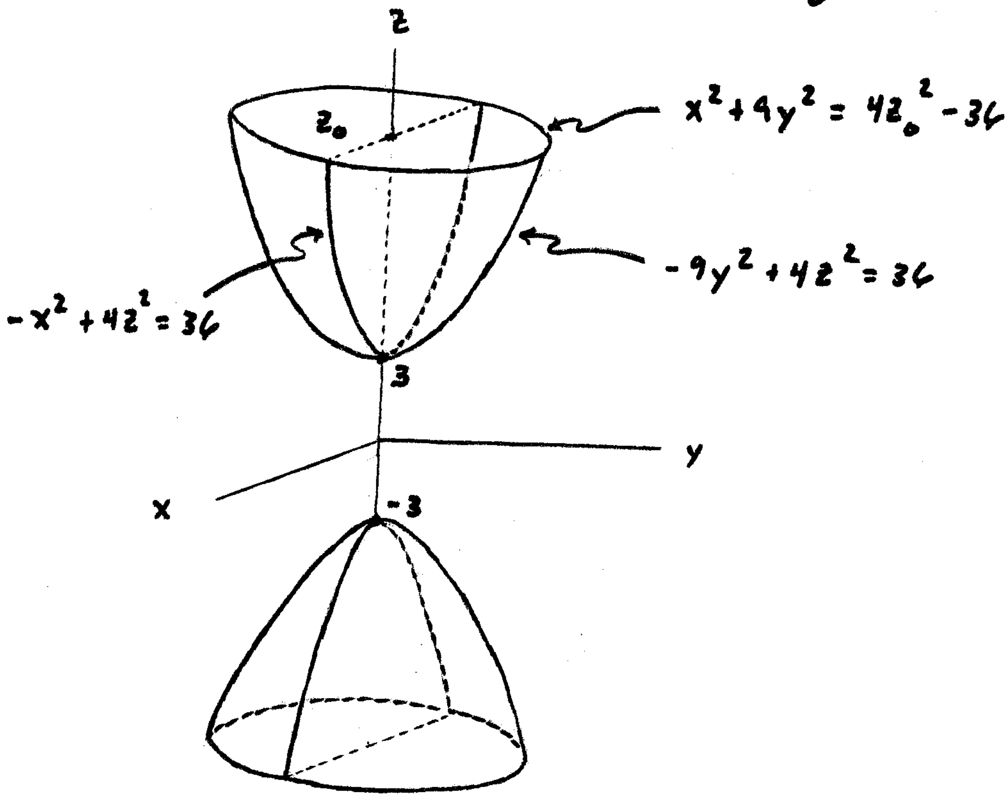
$x^2 + 9y^2 = -36$ EMPTY

$z = z_0: \quad -x^2 - 9y^2 = 36 - 4z_0^2$

$x^2 + 9y^2 = 4z_0^2 - 36$ EMPTY FOR $-3 < z_0 < 3$

A POINT FOR $z_0 = \pm 3$

ELLIPSE FOR $z_0 > 3$ AND $z_0 < -3$



HYPERBOLOID OF TWO SHEETS

SIMILARLY FOR ANY

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$\left(\text{OR} \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \quad \text{OR} \quad -\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \right)$$

JUST LIKE SPHERES, ALL OF THESE SURFACES CAN BE "TRANSLATED" ESSENTIALLY BY REPLACING x, y AND z B $x-x_0$, $y-y_0$ AND $z-z_0$.

<u>EXAMPLE</u> : $4x^2 + 4y^2 + z^2 + 8y - 4z + 4 = 0$

$$4x^2 + (4y^2 + 8y) + (z^2 - 4z) = -4$$

$$4x^2 + 4(y^2 + 2y \quad) + (z^2 - 4z \quad) = -4$$

$$4x^2 + 4(y^2 + 2y + 1) + (z^2 - 4z + 4) = -4 + 4 + 4$$

$$4(x-0)^2 + 4(y+1)^2 + (z-2)^2 = 4$$

$$\frac{(x-0)^2}{1^2} + \frac{(y-(-1))^2}{1^2} + \frac{(z-2)^2}{2^2} = 1$$

AND THIS IS THE ELLIPSOID $\frac{x^2}{1^2} + \frac{y^2}{1^2} + \frac{z^2}{2^2} = 1$ TRANSLATED FROM $(0,0,0)$ TO $(0,-1,2)$