

## DEFINITE INTEGRALS BY SUBSTITUTION

SUBSTITUTION FOR DEFINITE INTEGRALS, E.G.,

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{(3 + \sin x)^2} dx$$

TWO OPTIONS :

A. INDEFINITE INTEGRAL FIRST :

$$\begin{aligned} \int \frac{\cos x}{(3 + \sin x)^2} dx &= \int (3 + \sin x)^{-2} \cos x dx = \int u^{-2} du \\ &= \frac{u^{-2+1}}{-2+1} + C \\ &= -\frac{1}{u} + C \\ &= -\frac{1}{3 + \sin x} + C \end{aligned}$$

$u = 3 + \sin x$   
 $du = \cos x dx$

SELECT AN ANTIDERIVATIVE AND SUBSTITUTE LIMITS :

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \frac{\cos x}{(3 + \sin x)^2} dx &= -\frac{1}{3 + \sin x} \Big|_0^{\frac{\pi}{2}} \\ &= \left(-\frac{1}{3+1}\right) - \left(-\frac{1}{3+0}\right) = -\frac{1}{4} + \frac{1}{3} \\ &= \frac{1}{12} \end{aligned}$$

B. CHANGE  $x$ -LIMITS TO  $u$ -LIMITS WITH THE SUBSTITUTION :

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{(3+\sin x)^2} dx = \int_0^{\frac{\pi}{2}} (3+\sin x)^{-2} \cos x dx = \int_3^4 u^{-2} du$$

$$u = 3 + \sin x$$

$$du = \cos x dx$$

$$x=0 \Rightarrow u = 3 + \sin 0 = 3$$

$$x = \frac{\pi}{2} \Rightarrow u = 3 + \sin \frac{\pi}{2} = 4$$

$$= -\frac{1}{u} \Big|_3^4$$

$$= \left(-\frac{1}{4}\right) - \left(-\frac{1}{3}\right)$$

$$= \frac{1}{12}$$

MORE EXAMPLES : OPTION (B) IS GENERALLY FASTER SO WE'LL USE THIS.

$$1. \int_0^{\frac{\pi}{8}} \sin^2 2x \cos 2x dx = \frac{1}{2} \int_0^{\frac{\pi}{8}} \sin^2 2x (2 \cos 2x dx) =$$

$$u = \sin 2x$$

$$du = 2 \cos 2x dx$$

$$x=0 \Rightarrow u = \sin 2(0) = \sin 0 = 0$$

$$x = \frac{\pi}{8} \Rightarrow u = \sin 2\left(\frac{\pi}{8}\right) = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$\frac{1}{2} \int_0^{\frac{\sqrt{2}}{2}} u^2 du =$$

$$\frac{1}{2} \cdot \frac{1}{3} u^3 \Big|_0^{\frac{\sqrt{2}}{2}} =$$

$$\frac{1}{6} \left[ \left(\frac{\sqrt{2}}{2}\right)^3 - 0^3 \right] =$$

$$\frac{1}{6} \left[ \frac{\sqrt{2}}{4} \right] = \frac{\sqrt{2}}{24}$$

$$2. \int_1^{\sqrt{2}} \frac{x}{4+x^4} dx = \int_1^{\sqrt{2}} \frac{x}{4+(x^2)^2} dx = \int_1^{\sqrt{2}} \frac{x}{4(1+(\frac{x^2}{2})^2)} dx =$$

$$\frac{1}{4} \int_1^{\sqrt{2}} \frac{1}{1+(\frac{x^2}{2})^2} x dx = \frac{1}{4} \int_{\frac{1}{2}}^1 \frac{1}{1+u^2} du = \frac{1}{4} \arctan u \Big|_{\frac{1}{2}}^1$$

$$u = \frac{x^2}{2}$$

$$du = x dx$$

$$x=1 \Rightarrow u = \frac{1^2}{2} = \frac{1}{2}$$

$$x = \sqrt{2} \Rightarrow u = \frac{(\sqrt{2})^2}{2} = 1$$

$$= \frac{1}{4} [\arctan 1 - \arctan \frac{1}{2}]$$

$$= \frac{1}{4} \left[ \frac{\pi}{4} - \arctan \frac{1}{2} \right]$$

$$3. \int_4^7 \frac{dx}{x^2 - 6x + 9} = \int_4^7 \frac{1}{(x-3)^2} dx = \int_1^4 u^{-2} du = -\frac{1}{u} \Big|_1^4$$

$$u = x - 3 \quad = (-\frac{1}{4}) - (-\frac{1}{1})$$

$$du = dx \quad = \frac{3}{4}$$

$$x = 4 \Rightarrow u = 4 - 3 = 1$$

$$x = 7 \Rightarrow u = 7 - 3 = 4$$

$$4. \int_1^{\sqrt{e}} \frac{dx}{x \sqrt{1 - (\ln x)^2}} = \int_1^{\sqrt{e}} \frac{1}{\sqrt{1 - (\ln x)^2}} \frac{1}{x} dx = \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1 - u^2}} du =$$

$$u = \ln x \quad \text{ARCSIN } u \Big|_0^{\frac{1}{2}} =$$

$$du = \frac{1}{x} dx \quad \text{ARCSIN } \frac{1}{2} - \text{ARCSIN } 0 = \frac{\pi}{6} - 0$$

$$x = 1 \Rightarrow u = \ln 1 = 0 \quad = \frac{\pi}{6}$$

$$x = \sqrt{e} \Rightarrow u = \ln \sqrt{e} = \frac{1}{2}$$

$$5. \int_1^9 \frac{dx}{\sqrt{x}(1+x)} = \int_1^9 \frac{1}{1+x} \frac{1}{\sqrt{x}} dx = \int_1^9 \frac{1}{1+(\sqrt{x})^2} \frac{1}{\sqrt{x}} dx =$$

$$u = \sqrt{x}$$

$$du = \frac{1}{2\sqrt{x}} dx$$

$$x = 1 \Rightarrow u = \sqrt{1} = 1$$

$$x = 9 \Rightarrow u = \sqrt{9} = 3$$

$$2 \int_1^9 \frac{1}{1+(\sqrt{x})^2} \frac{1}{2\sqrt{x}} dx =$$

$$2 \int_1^3 \frac{1}{1+u^2} du = 2 \text{ARCTAN } u \Big|_1^3 = 2(\text{ARCTAN } 3 - \text{ARCTAN } 1)$$

$$= 2(\text{ARCTAN } 3 - \frac{\pi}{4})$$

$$\begin{aligned}
 6. \int_0^3 \frac{y \, dy}{\sqrt{y+1}} &= \int_1^4 \frac{u-1}{\sqrt{u}} \, du = \int_1^4 \left( \frac{u}{\sqrt{u}} - \frac{1}{\sqrt{u}} \right) du \\
 u = y+1 &\Rightarrow y = u-1 \\
 du = dy & \\
 y=0 &\Rightarrow u = 0+1 = 1 \\
 y=3 &\Rightarrow u = 3+1 = 4 \\
 &= \int_1^4 (u^{\frac{1}{2}} - u^{-\frac{1}{2}}) \, du \\
 &= \frac{2}{3} u^{\frac{3}{2}} \Big|_1^4 - 2u^{\frac{1}{2}} \Big|_1^4 \\
 &= \frac{2}{3} (4^{\frac{3}{2}} - 1^{\frac{3}{2}}) - 2(4^{\frac{1}{2}} - 1^{\frac{1}{2}}) \\
 &= \frac{2}{3} (8-1) - 2(2-1) \\
 &= \frac{14}{3} - 2 = \frac{8}{3}
 \end{aligned}$$

$$\begin{aligned}
 7. \int_0^1 \sqrt{e^x} \, dx &= \int_0^1 (e^x)^{\frac{1}{2}} \, dx = \int_0^1 e^{\frac{x}{2}} \, dx = 2 \int_0^1 e^{\frac{x}{2}} \left( \frac{1}{2} dx \right) = 2 \int_0^{\frac{1}{2}} e^u \, du \\
 u = \frac{x}{2} & \\
 du = \frac{1}{2} dx & \\
 x=0 &\Rightarrow u = \frac{0}{2} = 0 \\
 x=1 &\Rightarrow u = \frac{1}{2} \\
 &= 2e^u \Big|_0^{\frac{1}{2}} \\
 &= 2(e^{\frac{1}{2}} - e^0) \\
 &= 2(\sqrt{e} - 1)
 \end{aligned}$$