SOME MATHEMATICAL MODELS

1. RADIOACTIVE DECAY

RADIOACTIVE SUBSTANCES DECAY AT A RATE PROPORTIONAL TO THE AMOUNT OF THE SUBSTANCE PRESENT.

Q_0 = AMOUNT PRESENT INITIALLY (AT $t = 0$)

$Q(t)$ = AMOUNT PRESENT AT TIME t

FOR SOME POSITIVE CONSTANT k (THE DECAY RATE OF THE SUBSTANCE),

$$\begin{cases} \dfrac{dQ}{dt} = -kQ \\ Q(0) = Q_0 \end{cases}$$

SOLUTION :

$$\frac{dQ}{Q} = -k\,dt$$

$$\ln|Q| = -kt + C$$

$$\ln Q = -kt + C \qquad (Q(t) > 0)$$

$$Q(t) = e^{-kt} e^{C}$$

$$Q_0 = Q(0) = e^{C}$$

$$\boxed{Q(t) = Q_0 e^{-kt}}$$

QUESTION : HOW LONG BEFORE ONLY HALF OF THE INITIAL AMOUNT OF RADIOACTIVE SUBSTANCE REMAINS ?

$$Q(t) = \tfrac{1}{2} Q_0$$

$$Q_0 e^{-kt} = \tfrac{1}{2} Q_0$$

$$e^{-kt} = \tfrac{1}{2}$$

$$-kt = -\ln 2$$

$$\boxed{t = \frac{\ln 2}{k}}$$

THE ANSWER IS INDEPENDENT OF Q_0 AND CALLED THE HALF-LIFE OF THE RADIOACTIVE SUBSTANCE WITH DECAY RATE k.

NOTE : SIMILARLY, THE AMOUNT OF TIME REQUIRED FOR ANY GIVEN PERCENTAGE OF THE INITIAL AMOUNT TO DECAY DOES NOT DEPEND ON SIZE OF THE INITIAL AMOUNT. THIS IS THE BASIS FOR RADIOCARBON DATING.

2. SIMPLE POPULATION MODEL

SOME POPULATIONS (E.G., OF BACTERIA) GROW AT A RATE PROPORTIONAL TO THE SIZE OF THE POPULATION (AT LEAST, FOR A WHILE).

P_0 = INITIAL POPULATION

$P(t)$ = POPULATION AT TIME t

FOR SOME POSITIVE CONSTANT k (THE GROWTH RATE OF THE POPULATION)

$$\begin{cases} \dfrac{dP}{dt} = kP \\ P(0) = P_0 \end{cases}$$

SOLUTION : $P(t) = P_0 e^{kt}$

DOUBLING TIME : $t = \dfrac{\ln 2}{k}$

PROBLEM : THE POPULATION GROWS EXPONENTIALLY, FOREVER !

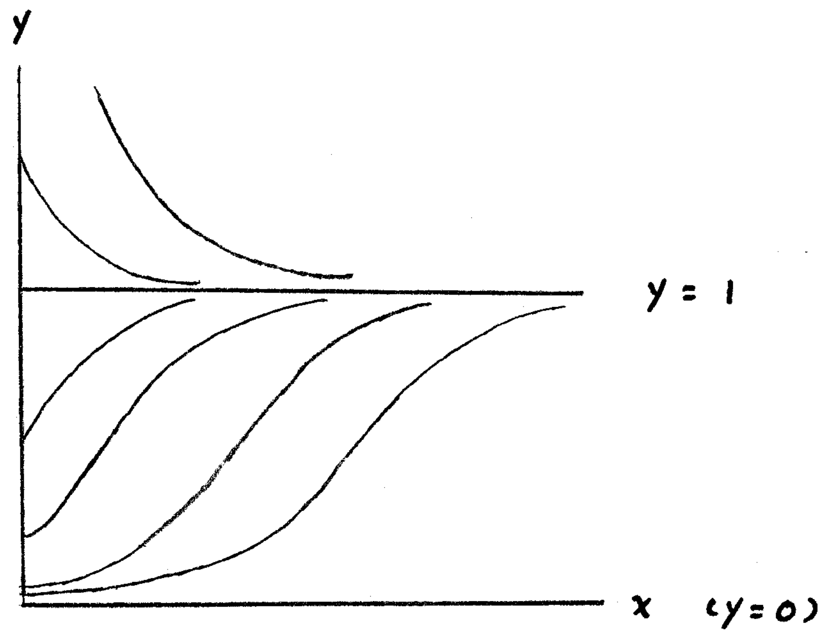
REAL POPULATIONS DON'T DO THIS, OF COURSE. A MORE ACCURATE MODEL WOULD CONTAIN TERMS THAT FIGHT THE GROWTH WHEN THE POPULATION BECOMES TOO LARGE, E.G.,

3. LOGISTIC MODELS

$$\begin{cases} \dfrac{dP}{dt} = kP(L-P) \qquad (= (kL)P - kP^2) \\ P(0) = P_0 \end{cases}$$

WHERE k AND L ARE POSITIVE CONSTANTS

EARLIER WE DREW A DIRECTION FIELD AND SOME INTEGRAL CURVES FOR A SPECIAL CASE : $y' = 4y(1-y)$



NOW WE SOLVE PROBLEM IN GENERAL.

$$\frac{dP}{dt} = kP(L-P)$$ NOTE : $P=0$ AND $P=L$ ARE SOLUTIONS

$$\frac{dP}{P(L-P)} = k\,dt$$ (ASSUMING $P \neq 0$ AND $P \neq L$)

$$\frac{1}{L}\left(\frac{1}{P} + \frac{1}{L-P}\right)dP = k\,dt$$ (PARTIAL FRACTIONS)

$$\ln|P| - \ln|L-P| = kLt + C$$

$$\ln\left|\frac{P}{L-P}\right| = kLt + C$$

$$\left|\frac{P}{L-P}\right| = e^{kLt}e^{C}$$

$$\frac{P}{L-P} = (\pm e^{c})e^{kLt}$$

$$\frac{P}{L-P} = Ae^{kLt}$$

AT $t=0$,

$$\frac{P_0}{L-P_0} = A$$

SO

$$\frac{P}{L-P} = \frac{P_0}{L-P_0}e^{kLt}$$

$$P = \left(\frac{P_0}{L-P_0}e^{kLt}\right)(L-P)$$

$$P = \frac{P_0 L}{L-P_0}e^{kLt} - \frac{P_0}{L-P_0}e^{kLt}P$$

$$\left(1+\frac{P_0}{L-P_0}e^{kLt}\right)P = \frac{P_0 L}{L-P_0}e^{kLt}$$

$$P(t) = \frac{\frac{P_0 L}{L-P_0}e^{kLt}}{1+\frac{P_0}{L-P_0}e^{kLt}} \qquad \left(\frac{\frac{L-P_0}{P_0}}{\frac{L-P_0}{P_0}}\right)$$

$$= \frac{Le^{kLt}}{\frac{L-P_0}{P_0}+e^{kLt}} \qquad \left(\frac{e^{-kLt}}{e^{-kLt}}\right)$$

$$P(t) = \frac{L}{1+\frac{L-P_0}{P_0}e^{-kLt}} \qquad \left(\frac{P_0}{P_0}\right)$$

$$\boxed{P(t) = \frac{P_0 L}{P_0+(L-P_0)e^{-kLt}}}$$

THINGS TO NOTICE ABOUT THIS POPULATION MODEL :

$P(t) = 0$ IS A SOLUTION (BORING)

$P(t) = L$ IS A SOLUTION (STABLE POPULATION)

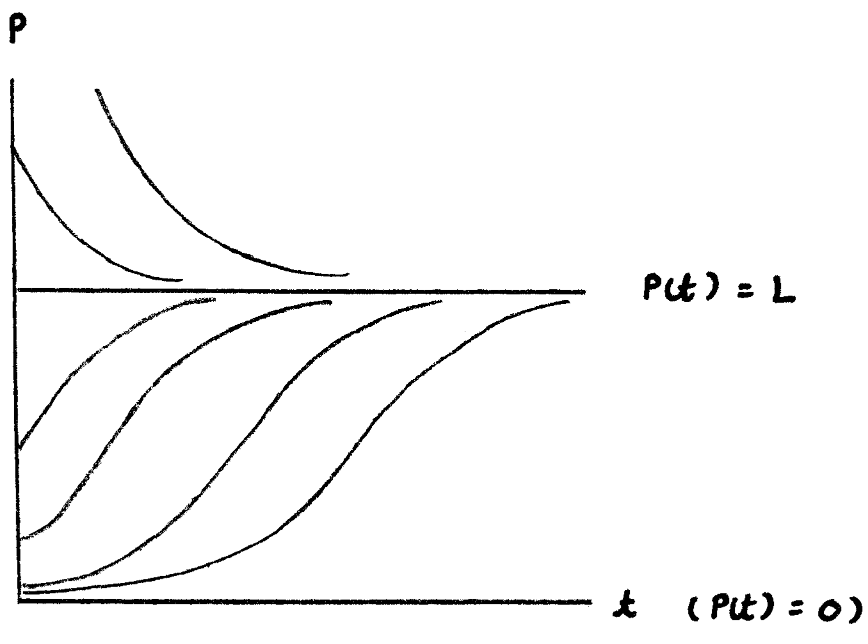
SINCE $kL > 0$, $e^{-kLt} \to 0$ AS $t \to \infty$ SO

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{P_0 L}{P_0 + (L - P_0)e^{-kLt}}$$

$$= \frac{P_0 L}{P_0 + 0}$$

$$= L$$

SO THE POPULATION APPROACHES ITS STABLE VALUE ASYMPTOTICALLY.



L IS CALLED THE <u>CARRYING</u> <u>CAPACITY</u> OF THE POPULATION.