

SERIES SOLUTIONS TO DIFFERENTIAL EQUATIONS

THE DIFFERENTIAL EQUATIONS WE HAVE SOLVED SO FAR ALL REQUIRED SOME SORT OF ALGEBRAIC "TRICK" FOR REWRITING THE EQUATION SO THAT WE CAN JUST INTEGRATE BOTH SIDES TO SOLVE (E.G., FOR LINEAR EQUATIONS SUCH AS $y' + 3y = e^{-2x}$, MULTIPLYING ON BOTH SIDES BY THE INTEGRATING FACTOR $\mu(x) = e^{3x}$ TURNS IT INTO $(e^{3x}y)' = e^x$, WHICH WE CAN INTEGRATE).

USUALLY SUCH TRICKS SIMPLY DON'T EXIST.

ALTERNATIVE : LOOK FOR A SOLUTION y GIVEN AS A POWER SERIES

$$y = \sum_{k=0}^{\infty} c_k x^k$$

BY JUST SUBSTITUTING THIS INTO THE EQUATION AND TRYING TO DETERMINE THE COEFFICIENTS c_0, c_1, c_2, \dots

WE WILL WARM UP WITH A FEW SIMPLE EXAMPLES THAT WE ALREADY KNOW HOW TO SOLVE AND THEN DO ONE THAT WE COULD NOT SOLVE ANY OTHER WAY.

EXAMPLES :

1. $y' - y = 0$

(PRETEND THAT YOU DON'T KNOW HOW TO SOLVE THIS.)

LET $y = \sum_{k=0}^{\infty} c_k x^k$. THEN

$$y' = \sum_{k=1}^{\infty} k c_k x^{k-1}$$

RE-INDEX : $n = k-1$ (SO $k = n+1$)
 $k = 1 \Rightarrow n = 0$

$$y' = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n$$

NOW SUBSTITUTE BOTH OF THESE (y AND y') INTO THE EQUATION $y' - y = 0$ (USING n FOR THE NAME OF THE SUMMATION INDEX IN BOTH) :

$$\sum_{n=0}^{\infty} (n+1) c_{n+1} x^n - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=0}^{\infty} [(n+1) c_{n+1} - c_n] x^n = 0$$

\Rightarrow

$$(n+1) c_{n+1} - c_n = 0 \quad \text{FOR ALL } n \geq 0$$

RECURRENCE RELATION : $c_{n+1} = \frac{1}{n+1} c_n$ FOR ALL $n \geq 0$

WRITE THESE OUT FOR THE FIRST FEW VALUES OF n :

$$n=0 : \quad c_1 = \frac{1}{1} c_0$$

$$n=1 : \quad c_2 = \frac{1}{2} c_1 = \frac{1}{2} \cdot \frac{1}{1} c_0$$

$$n=2 : \quad c_3 = \frac{1}{3} c_2 = \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{1} c_0$$

$$n=3 : \quad c_4 = \frac{1}{4} c_3 = \frac{1}{4} \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{1} c_0$$

$$\vdots$$

$$c_n = \frac{1}{n!} c_0 \quad \text{FOR } n \geq 0$$

THUS,

$$\begin{aligned} y &= \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{1}{n!} c_0 x^n \\ &= c_0 \sum_{n=0}^{\infty} \frac{1}{n!} x^n \end{aligned}$$

THERE IS NOTHING TO DETERMINE c_0 SO IT IS ARBITRARY. FOR ANY CHOICE OF c_0 THIS IS A SOLUTION TO THE EQUATION $y' - y = 0$.

NOTE : SINCE $\sum_{n=0}^{\infty} \frac{1}{n!} x^n = e^x$ THE ANSWER

IS THE SAME AS WE WOULD HAVE GOTTEN EARLIER :

$$y = c_0 e^x \quad (c_0 = \text{ARBITRARY CONSTANT})$$

$$2. \quad y'' + y = 0$$

(KEEP PRETENDING)

$$\text{LET } y = \sum_{k=0}^{\infty} c_k x^k. \text{ THEN}$$

$$y' = \sum_{k=1}^{\infty} k c_k x^{k-1}$$

$$y'' = \sum_{k=2}^{\infty} k(k-1) c_k x^{k-2}$$

RE-INDEX: $n = k-2$ (SO $k = n+2$ AND
 $k-1 = n+1$)

$$k=2 \Rightarrow n=0$$

$$y'' = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$$

NOW SUBSTITUTE BOTH y AND y'' INTO THE EQUATION $y'' + y = 0$

(USING n FOR THE NAME OF THE SUMMATION INDEX IN BOTH).

$$\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=0}^{\infty} [(n+2)(n+1) c_{n+2} + c_n] x^n = 0$$

$$\Rightarrow (n+2)(n+1) c_{n+2} + c_n = 0 \quad \text{FOR ALL } n \geq 0$$

RECURRENCE RELATION:

$$c_{n+2} = -\frac{1}{(n+2)(n+1)} c_n \quad \text{FOR ALL } n \geq 0$$

WRITE THESE OUT FOR THE FIRST FEW VALUES OF n :

$$n=0 : c_2 = -\frac{1}{2 \cdot 1} c_0$$

$$n=1 : c_3 = -\frac{1}{3 \cdot 2} c_1$$

$$n=2 : c_4 = -\frac{1}{4 \cdot 3} c_2 = \frac{1}{4 \cdot 3 \cdot 2 \cdot 1} c_0$$

$$n=3 : c_5 = -\frac{1}{5 \cdot 4} c_3 = \frac{1}{5 \cdot 4 \cdot 3 \cdot 2} c_1$$

⋮

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \dots$$

$$= c_0 + c_1 x - \frac{1}{2!} c_0 x^2 - \frac{1}{3!} c_1 x^3 + \frac{1}{4!} c_0 x^4 + \frac{1}{5!} c_1 x^5 - \dots$$

$$= c_0 \left[1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \dots \right] + c_1 \left[x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \dots \right]$$

$$= c_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} + c_1 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

THERE IS NOTHING TO DETERMINE c_0 AND c_1 , SO THEY ARE ARBITRARY.

FOR ANY CHOICE OF c_0 AND c_1 , THIS IS A SOLUTION TO $y'' + y = 0$

NOTE: SINCE $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = \cos x$ AND

$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = \sin x$, THE ANSWER IS THE

SAME AS WE WOULD HAVE GOTTEN EARLIER :

$$y = c_0 \cos x + c_1 \sin x$$

(c_0, c_1 , ARBITRARY)

3. $y'' - xy' - y = 0$ (NO NEED TO PRETEND. YOU DON'T KNOW HOW TO SOLVE THIS ONE.)

LET $y = \sum_{k=0}^{\infty} c_k x^k$. THEN

$$y' = \sum_{k=1}^{\infty} k c_k x^{k-1} \quad \text{AND} \quad xy' = x \sum_{k=1}^{\infty} k c_k x^{k-1} = \sum_{k=1}^{\infty} k c_k x^k$$

$$y'' = \sum_{k=2}^{\infty} k(k-1) c_k x^{k-2} = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$$

NOW SUBSTITUTE ALL OF THESE INTO THE EQUATION (USING n AS THE SUMMATION INDEX IN ALL OF THEM).

$$y'' - xy' - y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n - \sum_{n=1}^{\infty} n c_n x^n - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$(2c_2 - c_0) + \sum_{n=1}^{\infty} [(n+2)(n+1) c_{n+2} - n c_n - c_n] x^n = 0$$
$$- c_0 x^0 - \sum_{n=1}^{\infty} c_n x^n = 0$$

$$(2c_2 - c_0) + \sum_{n=1}^{\infty} [(n+2)(n+1) c_{n+2} - n c_n - c_n] x^n = 0$$

$$(2c_2 - c_0) + \sum_{n=1}^{\infty} [(n+2)(n+1) c_{n+2} - (n+1) c_n] x^n = 0$$

$$\Rightarrow 2c_2 - c_0 = 0$$

$$(n+2)(n+1) c_{n+2} - (n+1) c_n = 0 \quad \text{FOR ALL } n \geq 0$$

→

$$c_2 = \frac{1}{2} c_0$$

RECURRENCE RELATION : $c_{n+2} = \frac{1}{n+2} c_n$ FOR ALL $n \geq 0$

$$c_2 = \frac{1}{2} c_0$$

$$n = 1 : c_3 = \frac{1}{3} c_1$$

$$n = 2 : c_4 = \frac{1}{4} c_2 = \frac{1}{2 \cdot 4} c_0$$

$$n = 3 : c_5 = \frac{1}{5} c_3 = \frac{1}{3 \cdot 5} c_1$$

$$n = 4 : c_6 = \frac{1}{6} c_4 = \frac{1}{2 \cdot 4 \cdot 6} c_0$$

$$n = 5 : c_7 = \frac{1}{7} c_5 = \frac{1}{3 \cdot 5 \cdot 7} c_1$$

$$n = 6 : c_8 = \frac{1}{8} c_6 = \frac{1}{2 \cdot 4 \cdot 6 \cdot 8} c_0$$

⋮

$$\begin{aligned} y &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + c_7 x^7 + c_8 x^8 + \dots \\ &= c_0 + c_1 x + c_0 x^2 + \frac{1}{3} c_1 x^3 + \frac{1}{2 \cdot 4} c_0 x^4 + \frac{1}{3 \cdot 5} c_1 x^5 + \frac{1}{2 \cdot 4 \cdot 6} c_0 x^6 \\ &\quad + \frac{1}{3 \cdot 5 \cdot 7} c_1 x^7 + \frac{1}{2 \cdot 4 \cdot 6 \cdot 8} c_0 x^8 + \dots \\ &= c_0 \left[1 + \frac{1}{2} x^2 + \frac{1}{2 \cdot 4} x^4 + \frac{1}{2 \cdot 4 \cdot 6} x^6 + \frac{1}{2 \cdot 4 \cdot 6 \cdot 8} x^8 + \dots \right] \\ &\quad + c_1 \left[x + \frac{1}{3} x^3 + \frac{1}{3 \cdot 5} x^5 + \frac{1}{3 \cdot 5 \cdot 7} x^7 + \dots \right] \end{aligned}$$

NOTE THAT

$$\frac{1}{2 \cdot 4 \cdot 6 \cdot 8} = \frac{1}{(2 \cdot 1)(2 \cdot 2)(2 \cdot 3)(2 \cdot 4)} = \frac{1}{2^4 \cdot 4!}$$

$$\frac{1}{1 \cdot 3 \cdot 5 \cdot 7} = \frac{2 \cdot 4 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} = \frac{2^3 \cdot 3!}{7!}$$

AND SO

$$y = c_0 \left[\frac{1}{2^0 \cdot 0!} x^{2 \cdot 0} + \frac{1}{2^1 \cdot 1!} x^{2 \cdot 1} + \frac{1}{2^2 \cdot 2!} x^{2 \cdot 2} + \frac{1}{2^3 \cdot 3!} x^{2 \cdot 3} \right. \\ \left. + \frac{1}{2^4 \cdot 4!} x^{2 \cdot 4} + \dots \right]$$

$$+ c_1 \left[\frac{2^0 \cdot 0!}{1!} x^1 + \frac{2^1 \cdot 1!}{3!} x^3 + \frac{2^2 \cdot 2!}{5!} x^5 + \frac{2^3 \cdot 3!}{7!} x^7 + \dots \right]$$

$$= c_0 \sum_{k=0}^{\infty} \frac{1}{2^k \cdot k!} x^{2k} + c_1 \sum_{k=0}^{\infty} \frac{2^k \cdot k!}{(2k+1)!} x^{2k+1}$$

THERE IS NOTHING TO DETERMINE c_0 AND c_1 , SO THEY ARE ARBITRARY.

NOTE: THIS TIME THE FUNCTIONS $\sum_{k=0}^{\infty} \frac{1}{2^k k!} x^{2k}$

AND $\sum_{k=0}^{\infty} \frac{2^k k!}{(2k+1)!} x^{2k+1}$ CANNOT BE IDENTIFIED

WITH ANY SIMPLE COMBINATION OF ELEMENTARY

FUNCTIONS, BUT THEY ARE, NEVERTHELESS,

PERFECTLY FINE SOLUTIONS TO THE DIFFERENTIAL

EQUATION,

EXERCISES : FIND SERIES SOLUTIONS TO EACH OF THE FOLLOWING DIFFERENTIAL EQUATIONS. WRITE THE ANSWERS IN \sum -NOTATION.

1. $y' + y = 0$

2. $y'' + 4y = 0$

3. $(x-1)y' + 2y = 0$

4. $y'' + 2xy' + 2y = 0$

SOLUTIONS :

$$1. \quad y' + y = 0 \quad : \quad y = \sum_{k=0}^{\infty} c_k x^k$$

$$y' = \sum_{k=1}^{\infty} k c_k x^{k-1} = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n$$

$$\sum_{n=0}^{\infty} (n+1) c_{n+1} x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=0}^{\infty} [(n+1) c_{n+1} + c_n] x^n = 0 \quad \Rightarrow$$

$$(n+1)c_{n+1} + c_n = 0, \quad n \geq 0$$

$$c_{n+1} = -\frac{1}{n+1}c_n, \quad n \geq 0$$

$$n=0: \quad c_1 = -\frac{1}{1}c_0$$

$$n=1: \quad c_2 = -\frac{1}{2}c_1 = \frac{1}{1 \cdot 2}c_0$$

$$n=2: \quad c_3 = -\frac{1}{3}c_2 = -\frac{1}{1 \cdot 2 \cdot 3}c_0$$

$$n=3: \quad c_4 = -\frac{1}{4}c_3 = \frac{1}{1 \cdot 2 \cdot 3 \cdot 4}c_0$$

⋮

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \dots$$

$$= c_0 - \frac{1}{1}c_0x + \frac{1}{1 \cdot 2}c_0x^2 - \frac{1}{1 \cdot 2 \cdot 3}c_0x^3 + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4}c_0x^4 - \dots$$

$$= c_0 \left[\frac{1}{0!} - \frac{1}{1!}x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \dots \right]$$

$$= c_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k$$

NOTE: THIS IS $c_0 e^{-x}$

$$2. \quad y'' + 4y = 0: \quad y = \sum_{k=0}^{\infty} c_k x^k$$

$$y' = \sum_{k=1}^{\infty} k c_k x^{k-1}$$

$$y'' = \sum_{k=2}^{\infty} k(k-1)c_k x^{k-2}$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n + 4 \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=0}^{\infty} [(n+2)(n+1)c_{n+2} + 4c_n] x^n = 0 \Rightarrow$$

$$(n+2)(n+1)c_{n+2} + 4c_n = 0, \quad n \geq 0$$

$$c_{n+2} = -\frac{4}{(n+2)(n+1)} c_n, \quad n \geq 0$$

$$n=0 : \quad c_2 = -\frac{4}{2 \cdot 1} c_0$$

$$n=1 : \quad c_3 = -\frac{4}{3 \cdot 2} c_1$$

$$n=2 : \quad c_4 = -\frac{4}{4 \cdot 3} c_2 = \frac{4^2}{4 \cdot 3 \cdot 2 \cdot 1} c_0$$

$$n=3 : \quad c_5 = -\frac{4}{5 \cdot 4} c_3 = \frac{4^2}{5 \cdot 4 \cdot 3 \cdot 2} c_1$$

⋮

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \dots$$

$$= c_0 + c_1 x - \frac{4}{2!} c_0 x^2 - \frac{4}{3!} c_1 x^3 + \frac{4^2}{4!} c_0 x^4 + \frac{4^2}{5!} c_1 x^5 + \dots$$

$$= c_0 [1 - \frac{4}{2!} x^2 + \frac{4^2}{4!} x^4 - \dots]$$

$$+ c_1 [x - \frac{4}{3!} x^3 + \frac{4^2}{5!} x^5 - \dots]$$

$$= c_0 \sum_{k=0}^{\infty} \frac{(-1)^k 4^k}{(2k)!} x^{2k} + c_1 \sum_{k=0}^{\infty} \frac{(-1)^k 4^k}{(2k+1)!} x^{2k+1}$$

$$= c_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} (2x)^{2k} + \frac{c_1}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (2x)^{2k+1}$$

NOTE : THIS IS $c_0 \cos(2x) + \frac{c_1}{2} \sin(2x)$

$$3. (x-1)y' + 2y = 0 : y = \sum_{k=0}^{\infty} c_k x^k$$

$$y' = \sum_{k=1}^{\infty} k c_k x^{k-1}$$

$$(x-1)y' = (x-1) \sum_{k=1}^{\infty} k c_k x^{k-1}$$

$$= x \sum_{k=1}^{\infty} k c_k x^{k-1} - \sum_{k=1}^{\infty} k c_k x^{k-1}$$

$$= \sum_{k=1}^{\infty} k c_k x^k - \sum_{k=1}^{\infty} k c_k x^{k-1}$$

$$= \sum_{n=1}^{\infty} n c_n x^n - \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n$$

$$\sum_{n=1}^{\infty} n c_n x^n - \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n + 2 \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=1}^{\infty} n c_n x^n - (0+1) c_{0+1} x^0 - \sum_{n=1}^{\infty} (n+1) c_{n+1} x^n + 2 c_0 x^0$$

$$+ \sum_{n=1}^{\infty} 2 c_n x^n = 0$$

$$(-c_1 + 2c_0) + \sum_{n=1}^{\infty} [n c_n - (n+1) c_{n+1} + 2c_n] x^n = 0$$

$$(-c_1 + 2c_0) + \sum_{n=1}^{\infty} [(n+2) c_n - (n+1) c_{n+1}] x^n = 0$$

⇒

$$c_1 = 2c_0$$

$$c_{n+1} = \frac{n+2}{n+1} c_n, \quad n \geq 1$$

$$n = 1: \quad c_2 = \frac{3}{2} c_1 = (2) \frac{3}{2} c_0$$

$$n = 2: \quad c_3 = \frac{4}{3} c_2 = (2) \frac{4}{3} \cdot \frac{3}{2} c_0$$

$$n = 3: \quad c_4 = \frac{5}{4} c_3 = (2) \frac{5}{4} \cdot \frac{4}{3} \cdot \frac{3}{2} c_0$$

$$n = 4: \quad c_5 = \frac{6}{5} c_4 = (2) \frac{6}{5} \cdot \frac{5}{4} \cdot \frac{4}{3} \cdot \frac{3}{2} c_0$$

CANCELLING GIVES

$$c_1 = 2c_0$$

$$c_2 = 3c_0$$

$$c_3 = 4c_0$$

$$c_4 = 5c_0$$

$$c_5 = 6c_0$$

SO

$$y = c_0 [1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + \dots]$$

$$= c_0 \sum_{k=0}^{\infty} (k+1) x^k$$

$$4. \quad y'' + 2xy' + 2y = 0$$

$$y = \sum_{k=0}^{\infty} c_k x^k$$

$$y' = \sum_{k=1}^{\infty} k c_k x^{k-1} \quad \text{and} \quad 2xy' = \sum_{k=1}^{\infty} 2k c_k x^k$$

$$y'' = \sum_{k=2}^{\infty} k(k-1) c_k x^{k-2} = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n + \sum_{n=1}^{\infty} 2n c_n x^n + \sum_{n=0}^{\infty} 2c_n x^n = 0$$

$$(0+2)(0+1) c_{0+2} x^0 + 2c_0 x^0 +$$

$$\sum_{n=1}^{\infty} [(n+2)(n+1) c_{n+2} + 2(n+1) c_n] x^n = 0$$

$$\Rightarrow \quad 2c_2 + 2c_0 = 0$$

$$(n+2)(n+1) c_{n+2} + 2(n+1) c_n = 0, \quad n \geq 1$$

$$\Rightarrow \quad c_2 = -c_0$$

$$\Rightarrow \quad c_{n+2} = -\frac{2}{n+2} c_n, \quad n \geq 0$$

$$n=1: \quad c_3 = -\frac{2}{3} c_1$$

$$n=2: \quad c_4 = -\frac{2}{4} c_2 = -\frac{2}{4} (-c_0) = \frac{(-2)^1}{4} c_0$$

$$n=3: \quad c_5 = -\frac{2}{5} c_3 = -\frac{2}{5} \left(-\frac{2}{3} c_1\right) = \frac{(-2)^2}{3 \cdot 5} c_1$$

$$n=4 : c_6 = -\frac{2}{6} c_4 = -\frac{2}{6} \left(\frac{(-2)^1}{4} c_0 \right) = \frac{(-2)^2}{4 \cdot 6} c_0$$

$$n=5 : c_7 = -\frac{2}{7} c_5 = -\frac{2}{7} \left(\frac{(-2)^2}{3 \cdot 5} c_1 \right) = \frac{(-2)^3}{3 \cdot 5 \cdot 7} c_1$$

$$n=6 : c_8 = -\frac{2}{8} c_6 = -\frac{2}{8} \left(\frac{(-2)^2}{4 \cdot 6} c_0 \right) = \frac{(-2)^3}{4 \cdot 6 \cdot 8} c_0$$

EXCEPT FOR THE FACTORS OF -2 THE PATTERN HERE IS EXACTLY THE SAME AS IN EXAMPLE 3, PAGE 8.

$$y = c_0 \left[1 + \frac{(-2)^1}{2^1 \cdot 1!} x^2 + \frac{(-2)^2}{2^2 \cdot 2!} x^4 + \frac{(-2)^3}{2^3 \cdot 3!} x^6 + \frac{(-2)^4}{2^4 \cdot 4!} x^8 + \dots \right]$$

$$+ c_1 \left[\frac{(-2)^0 \cdot 2^0 \cdot 0!}{1!} x^1 + \frac{(-2)^1 \cdot 2^1 \cdot 1!}{3!} x^3 + \frac{(-2)^2 \cdot 2^2 \cdot 2!}{5!} x^5$$

$$+ \frac{(-2)^3 \cdot 2^3 \cdot 3!}{7!} x^7 + \dots \right]$$

$$= c_0 \left[1 + \frac{(-1)^1 \cdot 2}{2^1 \cdot 1!} x^2 + \frac{(-1)^2 \cdot 2^2}{2^2 \cdot 2!} x^4 + \frac{(-1)^3 \cdot 2^3}{2^3 \cdot 3!} x^6 + \frac{(-1)^4 \cdot 2^4}{2^4 \cdot 4!} x^8 + \dots \right]$$

$$+ c_1 \left[\frac{(-1)^0 \cdot 2^0 \cdot 2^0 \cdot 0!}{1!} x^1 + \frac{(-1)^1 \cdot 2^1 \cdot 2^1 \cdot 1!}{3!} x^3 + \frac{(-1)^2 \cdot 2^2 \cdot 2^2 \cdot 2!}{5!} x^5$$

$$+ \frac{(-1)^3 \cdot 2^3 \cdot 2^3 \cdot 3!}{7!} x^7 + \dots \right]$$

$$= c_0 \left[\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^{2k} \right]$$

$$+ c_1 \sum_{k=0}^{\infty} \frac{(-1)^k \cdot 2^{2k} \cdot k!}{(2k+1)!} x^{2k+1}$$