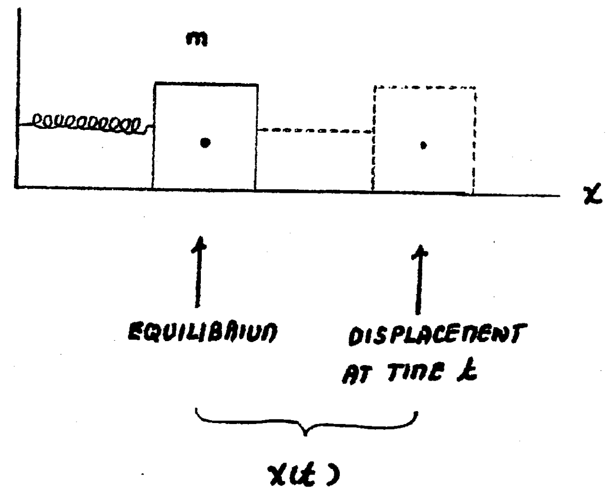
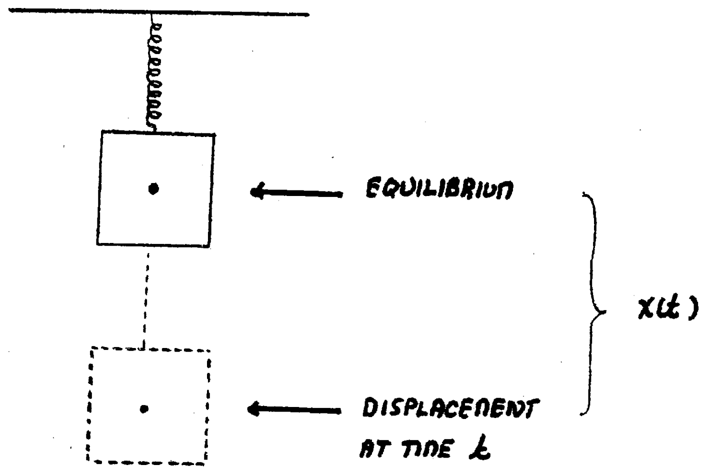
SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS :

MOTIVATION : CONSIDER A MASS ON A SPRING, E.G.,



OR MAYBE



TO DESCRIBE THE MOTION WE NEED TO FIND $x(t)$.

NEWTON'S SECOND LAW :

$$F = mA$$

$$\underbrace{-kx}_{\text{SPRING}} \underbrace{- c\frac{dx}{dt}}_{\text{DAMPING}} \underbrace{+ f(t)}_{\text{EXTERNAL APPLIED FORCE}} = m\frac{d^2x}{dt^2}$$

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = f(t)$$

SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS :

$$y'' + p(x)y' + q(x)y = r(x) \qquad (\text{NONHOMOGENEOUS})$$

$$y'' + p(x)y' + q(x)y = 0 \qquad (\text{HOMOGENEOUS})$$

WHERE $p(x)$, $q(x)$ AND $r(x)$ ARE KNOWN FUNCTIONS OF X.

WE WILL CONSIDER ONLY THE HOMOGENEOUS CASE.

<u>EXAMPLE</u> : $y'' + y = 0$ $\qquad (y'' + 0y' + 1y = 0)$

WE HAVE ALREADY FOUND LOTS OF SOLUTIONS BY INSPECTION :

$$\sin x$$
$$\cos x$$
$$A \sin x$$
$$B \cos x$$
$$A \sin x + B \cos x$$

(A, B ARE ARBITRARY CONSTANTS)

IT WOULD BE NICE TO KNOW THAT WE HAVE ACTUALLY FOUND THEM ALL.

<u>THEOREM</u> : SUPPOSE $p(x)$ AND $q(x)$ ARE CONTINUOUS ON SOME OPEN INTERVAL I. THEN THE EQUATION

$$y'' + p(x)\,y' + q(x)\,y = 0$$

HAS (INFINITELY MANY) SOLUTIONS ON I. MOREOVER, IF $y_1(x)$ AND $y_2(x)$ ARE TWO SOLUTIONS AND IF NEITHER OF THESE IS A CONSTANT MULTIPLE OF THE OTHER (WE SAY THAT $y_1(x)$ AND $y_2(x)$ ARE <u>LINEARLY</u> <u>INDEPENDENT</u>), THEN <u>EVERY</u> SOLUTION CAN BE WRITTEN IN THE FORM

$$C_1\, y_1(x) + C_2\, y_2(x)$$

FOR SOME CONSTANTS C_1 AND C_2 .

THE PROOF OF THIS IS TOO HARD FOR US IN HERE SO WE WILL NOT DISCUSS IT. WE WILL USE IT, HOWEVER.

<u>EXAMPLES</u>:

1. $y'' + y = 0$

   $y_1(x) = \sin x$ AND $y_2(x) = \cos x$ ARE TWO SOLUTIONS. THEY ARE LINEARLY INDEPENDENT BECAUSE $\sin x$ IS NOT A CONSTANT MULTIPLE OF $\cos x$ (IF IT WERE TRUE THAT $\sin x = C \cos x$ FOR ALL x, THEN $\sin 0 = C \cos 0$ SO $0 = C$, AND SO $\sin x = 0$ FOR ALL x AND THAT'S JUST SILLY) AND $\cos x$ IS NOT A CONSTANT MULTIPLE OF $\sin x$ (SIMILAR ARGUMENT). THUS, EVERY SOLUTION TO $y'' + y = 0$ IS OF THE FORM

   $$c_1 \sin x + c_2 \cos x$$

   FOR SOME CONSTANTS c_1 AND c_2.

2. $y'' - y = 0$

   $y_1(x) = e^x$ IS A SOLUTION $\left((e^x)'' - e^x = e^x - e^x = 0 \right)$ AND SO IS $y_2(x) = e^{-x}$ $\left((e^{-x})'' - e^{-x} = e^{-x} - e^{-x} = 0 \right)$ MOREOVER, THESE ARE LINEARLY INDEPENDENT :

$e^{x} = ce^{-x} \Rightarrow e^{2x} = c$ BUT THIS CAN'T BE SINCE e^{2x} IS 1 AT $x=0$ AND e AT $x = \frac{1}{2}$.

SIMILARLY, $e^{-x} = ce^{x}$ IS IMPOSSIBLE. THUS, EVERY SOLUTION TO $y'' - y = 0$ IS

$$c_1 e^{x} + c_2 e^{-x}$$

FOR SOME CONSTANTS c_1 AND c_2.

3. $y'' - 2y' + y = 0$

NOTICE THAT $y_1(x) = e^{x}$ IS A SOLUTION.

$$(e^{x})'' - 2(e^{x})' + (e^{x}) = e^{x} - 2e^{x} + e^{x} = 0.$$

A SECOND SOLUTION THAT ISN'T A CONSTANT MULTIPLE OF e^{x} ISN'T SO EASY TO SPOT, BUT A LITTLE EXPERIMENTATION GIVES $y_2(x) = xe^{x}$.

$$\begin{aligned}(xe^{x})'' - 2(xe^{x})' + xe^{x} &= (xe^{x} + e^{x})' - 2(xe^{x} + e^{x}) + xe^{x} \\ &= xe^{x} + e^{x} + e^{x} - 2xe^{x} - 2e^{x} + xe^{x} \\ &= 0\end{aligned}$$

THESE ARE LINEARLY INDEPENDENT BECAUSE

$$e^x = cxe^x \quad \text{IMPLIES} \quad e^0 = c(0)e^0$$
$$1 = 0$$

WHICH IS STUPID

$$xe^x = ce^x \quad \text{IMPLIES} \quad 0e^0 = ce^0$$
$$0 = c$$

SO $xe^x = 0$ AND

$$1e^1 = 0$$
$$e = 0$$

AND THIS IS JUST AS BAD

THUS, EVERY SOLUTION TO $y'' - 2y' + y = 0$ IS

$$c_1 e^x + c_2 x e^x$$

FOR SOME CONSTANTS c_1 AND c_2 .

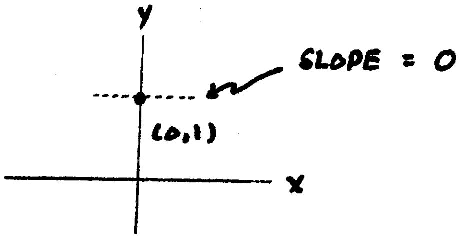
NOTICE THAT THE SOLUTIONS TO THESE EQUATIONS CONTAIN <u>TWO</u> ARBITRARY CONSTANTS AND SPECIFYING A POINT FOR THE SOLUTION TO GO THROUGH ($y(x_0) = y_0$) IS ONLY <u>ONE</u> CONDITION SO IT CAN'T DETERMINE BOTH CONSTANTS

FOR SECOND ORDER LINEAR EQUATIONS A UNIQUE SOLUTION IS DETERMINED BY SPECIFYING

1. A POINT FOR IT TO GO THROUGH, AND
2. THE SLOPE WITH WHICH IT SHOULD GO THROUGH THIS POINT.

EXAMPLE : SOLVE THE INITIAL VALUE PROBLEM

$$\begin{cases} y'' - y = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$$



WE ALREADY KNOW THE SOLUTIONS TO $y'' - y = 0$:

$$y(x) = c_1 e^{x} + c_2 e^{-x}$$

$y(0) = 1 \Rightarrow c_1 e^{0} + c_2 e^{-0} = 1 \Rightarrow c_1 + c_2 = 1$

$y'(x) = c_1 e^{x} - c_2 e^{-x}$ SO

$y'(0) = 0 \Rightarrow c_1 e^{0} - c_2 e^{-0} = 0 \Rightarrow c_1 - c_2 = 0$

THUS, WE NEED

$$\begin{cases} c_1 + c_2 = 1 \\ c_1 - c_2 = 0 \end{cases}$$

ADD

$$2c_1 = 1$$

$$c_1 = \frac{1}{2}$$

THEN

$$c_2 = c_1 = \frac{1}{2}$$

SOLUTION : $y(x) = \frac{1}{2} e^{x} + \frac{1}{2} e^{-x} = \frac{1}{2}(e^{x} + e^{-x}) = \cosh x$

EQUATIONS LIKE $y'' + p(x)y' + q(x)y = 0$ ARE GENERALLY IMPOSSIBLE TO SOLVE EXPLICITLY <u>EXCEPT</u> WHEN THE COEFFICIENTS ARE CONSTANT.

2^{ND} ORDER, LINEAR, HOMOGENEOUS EQUATIONS WITH CONSTANT COEFFICIENTS :

$$\boxed{ay'' + by' + cy = 0}$$

(a, b, c CONSTANTS AND $a \neq 0$)

<u>NOTE</u> : SOME THINGS TO FOLLOW WILL LOOK MORE FAMILIAR IF WE DO NOT INSIST THAT THE COEFFICIENT OF y'' IS 1.

WE'RE GOING TO BEGIN WITH THE "GUESS" THAT SUCH AN EQUATION SHOULD HAVE EXPONENTIAL SOLUTIONS, I.E., WE WILL LOOK FOR SOLUTIONS OF THE FORM

$$y = e^{rx}$$

COMPUTE THE DERIVATIVES, PLUG INTO $ay'' + by' + cy = 0$ AND TRY SOLVE FOR r :

$$y = e^{rx}$$
$$y' = re^{rx}$$
$$y'' = r^2 e^{rx}$$

$$ay'' + by' + cy = ar^2e^{rx} + bre^{rx} + ce^{rx}$$
$$= (ar^2 + br + c)e^{rx}$$

WHICH CAN ONLY BE ZERO IF

$$\boxed{ar^2 + br + c = 0}$$

(CALLED THE <u>CHARACTERISTIC</u> <u>EQUATION</u>, OR <u>AUXILIARY</u> <u>EQUATION</u>, FOR $ay'' + by' + cy = 0$).

THE SOLUTIONS

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

MAY BE

1. REAL AND DISTINCT
2. REAL AND REPEATED
3. COMPLEX

1. IF THE SOLUTIONS TO THE CHARACTERISTIC EQUATION ARE TWO DISTINCT REAL NUMBERS r_1 AND r_2, THEN $e^{r_1 x}$ AND $e^{r_2 x}$ ARE SOLUTIONS TO THE DIFFERENTIAL EQUATION. THEY ARE LINEARLY INDEPENDENT (AN EXERCISE FOR YOU) SO

$$\boxed{c_1 e^{r_1 x} + c_2 e^{r_2 x}}$$

IS THE GENERAL SOLUTION TO THE DIFFERENTIAL EQUATION.

EXAMPLE : $y'' - y' - 6y = 0$

CHARACTERISTIC EQUATION :

$$r^2 - r - 6 = 0$$

$$(r-3)(r+2) = 0$$

$$r = -2, 3$$

SOLUTION TO DIFFERENTIAL EQUATION :

$$c_1 e^{-2x} + c_2 e^{3x}$$

2. IF THERE IS JUST ONE REAL SOLUTION r TO THE CHARACTERISTIC EQUATION, THEN e^{rx} IS A SOLUTION TO THE DIFFERENTIAL EQUATION. AS IN EXAMPLE # 3, PAGE 5, A SECOND LINEARLY INDEPENDENT SOLUTION IS xe^{rx} SO

$$\boxed{c_1 e^{rx} + c_2 x e^{rx}}$$

IS THE GENERAL SOLUTION TO THE DIFFERENTIAL EQUATION.

EXAMPLE : $y'' - 8y' + 16y = 0$

CHARACTERISTIC EQUATION :

$$r^2 - 8r + 16 = 0$$
$$(r-4)(r-4) = 0$$
$$r = 4$$

SOLUTION TO THE DIFFERENTIAL EQUATION :

$$c_1 e^{4x} + c_2 x e^{4x}$$

3. IF THE SOLUTIONS TO THE CHARACTERISTIC EQUATION ARE COMPLEX

$$r = a \pm bi$$

THEN THE GENERAL SOLUTION TO THE DIFFERENTIAL EQUATION IS

$$\boxed{c_1 e^{ax} \cos bx + c_2 e^{ax} \sin bx}$$

NOTE : ADMITTEDLY, THIS LOOKS A LITTLE WEIRD. IT WILL HELP TO UNDERSTAND WHERE THESE FUNCTIONS CAME FROM IF YOU REMEMBER SOMETHING ABOUT COMPLEX NUMBERS :

EULER'S FORMULA : $e^{ti} = \cos t + i \sin t$

(WE'LL DERIVE THIS USING TAYLOR SERIES IN A FEW WEEKS.)

WITH THIS WE CAN COMPUTE

$$e^{rx} = e^{(a+bi)x} = e^{ax+bxi} = e^{ax} e^{bxi}$$

$$= e^{ax} (\cos bx + i \sin bx)$$

$$= e^{ax} \cos bx + i e^{ax} \sin bx$$

NOW YOU CAN AT LEAST SEE WHERE THE FUNCTIONS $e^{ax} \cos bx$ AND $e^{ax} \sin bx$ CAME FROM. JUST PLUGGING THEM INTO THE EQUATION SHOWS THAT THEY ARE SOLUTIONS AND IT IS EASY TO SEE THAT THEY ARE LINEARLY INDEPENDENT.

<u>EXAMPLE</u> : $y'' + y' + y = 0$

CHARACTERISTIC EQUATION :

$$r^2 + r + 1 = 0$$

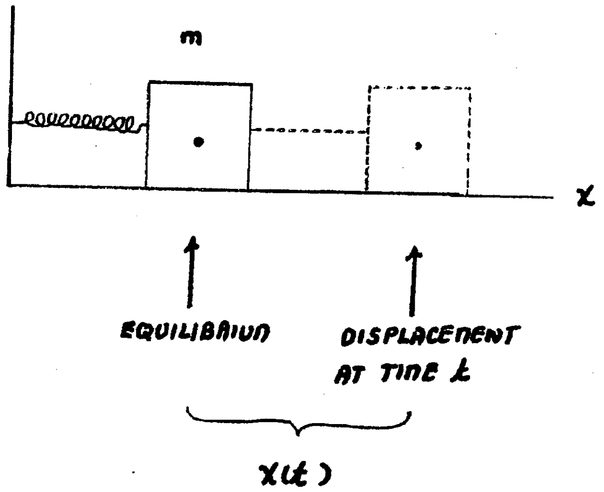
$$r = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} = \frac{-1 \pm \sqrt{3}\, i}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$$a = -\frac{1}{2}$$
$$b = \frac{\sqrt{3}}{2}$$

SOLUTION TO THE DIFFERENTIAL EQUATION :

$$c_1 e^{-\frac{1}{2}x} \cos \left(\frac{\sqrt{3}}{2} x \right) + c_2 e^{-\frac{1}{2}x} \sin \left(\frac{\sqrt{3}}{2} x \right)$$

NOW WE WILL RETURN TO THE EXMPLE THAT MOTIVATED ALL OF THIS IN THE FIRST PLACE.



$$m \frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

MASS (POSITIVE)

DAMPING CONSTANT ($c \geq 0$)

SPRING CONSTANT (POSITIVE)

NO EXTERNAL APPLIED FORCES

$x(0)$ = INITIAL DISPLACEMENT FROM EQUILIBRIUM

$x'(0)$ = INITIAL VELOCITY

FIRST SUPPOSE THAT $c = 0$ (NO DAMPING, I.E., FRICTIONLESS PLANE, NO ATMOSPHERE, ETC.)

$$m \frac{d^2x}{dt^2} + kx = 0$$

CHARACTERISTIC EQUATION :

$$m r^2 + k = 0$$

$$r^2 = -\frac{k}{m}$$

$$r = \pm \sqrt{\frac{k}{m}}\, i$$

SOLUTION :

$$x(t) = c_1 \cos \sqrt{\frac{k}{m}}\, t + c_2 \sin \sqrt{\frac{k}{m}}\, t$$

INTRODUCE

$$\omega_0 = \sqrt{\frac{k}{m}}$$ = <u>NATURAL</u> <u>FREQUENCY</u> OF THE SYSTEM

$$\boxed{x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t}$$

c_1 AND c_2 ARE DETERMINED FROM THE INITIAL CONDITIONS.

A MORE CONVENIENT WAY OF WRITING THE SOLUTION :

$$\boxed{x(t) = R \cos(\omega_0 t - \delta)}$$

WHERE R IS THE <u>AMPLITUDE</u> OF THE VIBRATION AND δ IS THE <u>PHASE</u> <u>SHIFT</u>.

HERE IS HOW TO DETERMINE R AND δ :

$$x(t) = R\cos(\omega_0 t - \delta)$$

$$= R\cos\omega_0 t\cos\delta + R\sin\omega_0 t\sin\delta$$

$$= (R\cos\delta)\cos\omega_0 t + (R\sin\delta)\sin\omega_0 t$$

AND

$$x(t) = c_1\cos\omega_0 t + c_2\sin\omega_0 t$$

IMPLY THAT WE MUST HAVE

$$c_1 = R\cos\delta \quad \text{AND} \quad c_2 = R\sin\delta \, .$$

THIS IMPLIES

$$c_1^2 + c_2^2 = R^2\cos^2\delta + R^2\sin^2\delta$$

$$= R^2(\cos^2\delta + \sin^2\delta)$$

$$= R^2$$

SO, TAKING R TO BE POSITIVE,

$$\boxed{R = \sqrt{c_1^2 + c_2^2}}$$

NOW THAT WE KNOW c_1, c_2 AND R, FINDING δ IS JUST TRIGONOMETRY:

$$\cos\delta = \frac{c_1}{R} \quad \text{AND} \quad \sin\delta = \frac{c_2}{R}$$

(OF COURSE, δ IS DETERMINED ONLY UP TO MULTIPLES OF 2π)

FROM

$$x(t) = R\cos(\omega_0 t - \delta)$$

THE CHARACTERISTICS OF THE MOTION IN THIS CASE ARE CLEAR :

PERIODIC WITH PERIOD

$$\frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}} \quad .$$

MAXIMAL DISPLACEMENT FROM EQUILIBRIUM R.

NOW WE'LL ADD DAMPING, I.E., TAKE $c > 0$.

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

CHARACTERISTIC EQUATION :

$$mr^2 + cr + k = 0$$

$$r = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

WE WILL CONSIDER THREE CASES.

1. $c^2 - 4mk > 0$, I.E., $c > 2\sqrt{mk}$

2. $c^2 - 4mk = 0$, I.E., $c = 2\sqrt{mk}$

3. $c^2 - 4mk < 0$, I.E., $0 < c < 2\sqrt{mk}$

IF $c^2 - 4mk > 0$ THERE ARE TWO DISTINCT, <u>NEGATIVE</u> SOLUTIONS

$$r_1 = \frac{-c + \sqrt{c^2 - 4mk}}{2m} \quad \text{AND} \quad r_2 = \frac{-c - \sqrt{c^2 - 4mk}}{2m}$$

SO

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} .$$

MOTION IS <u>NOT</u> OSCILLATORY : INITIAL DISPLACEMENT OF

$$X(0) = c_1 + c_2$$

AND

$$x(t) \longrightarrow 0 \quad \text{AS} \quad t \rightarrow 0 .$$

THE SYSTEM IS SAID TO BE <u>OVERDAMPED</u> (TOO MUCH DAMPING TO PERMIT OSCILLATION).

IF $c^2 - 4mk = 0$, THEN THERE IS ONE <u>NEGATIVE</u> SOLUTION

$$r = -\frac{c}{2m}$$

SO

$$x(t) = c_1 e^{-ct/2m} + c_2 t e^{-ct/2m}$$

$$= (c_1 + c_2 t) e^{-ct/2m}$$

MOTION IS SIMILAR TO THE LAST CASE. INITIAL DISPLACEMENT OF $X(0) = c_1$ AND $x(t) \rightarrow 0$ AS $t \rightarrow 0$. THE SYSTEM IS

SAID TO BE <u>CRITICALLY</u> <u>DAMPED</u> ($c = 2\sqrt{mk}$ IS STILL TOO MUCH DAMPING TO PERMIT OSCILLATIONS, BUT, AS WE WILL NOW SHOW, IT IS "ON THE EDGE" IN THE SENSE THAT <u>ANY</u> DECREASE IN c WILL GIVE OSCILLATORY SOLUTIONS).

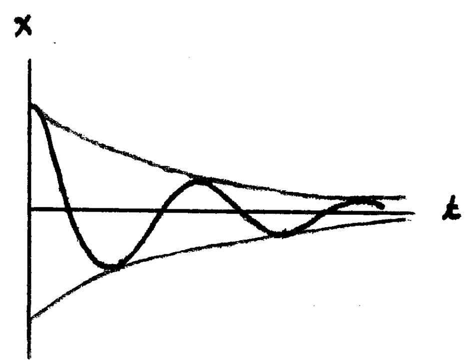
FINALLY, SUPPOSE $c^2 - 4mk < 0$ SO THAT THE SOLUTIONS ARE COMPLEX

$$-\frac{c}{2m} \pm \frac{\sqrt{4mk - c^2}}{2m}\, i = \alpha \pm \beta i .$$

THUS,

$$\begin{aligned} x(t) &= c_1 e^{\alpha t} \cos \beta t + c_2 e^{\alpha t} \sin \beta t \\ &= (c_1 \cos \beta t + c_2 \sin \beta t) e^{\alpha t} \\ &= R \cos (\beta t - \delta) e^{\alpha t} \end{aligned}$$

NOW THE MOTION IS, INDEED, OSCILLATORY (NOT PERIODIC) AND "DAMPS OUT" ($x(t) \to 0$) AS $t \to \infty$.



THINGS GET MORE INTERESTING WHEN THERE ARE ALSO EXTERNAL APPLIED FORCES, BUT THIS WILL HAVE TO WAIT.