POWER SERIES

RECALL :

GIVEN A FUNCTION $f(x)$ AND A POINT x_0 IN ITS DOMAIN, THE BEST n^{TH} DEGREE POLYNOMIAL APPROXIMATION TO $f(x)$ NEAR x_0 IS THE n^{TH} TAYLOR POLYNOMIAL FOR $f(x)$ AT x_0 :

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

$$= f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

$x_0 = 0$ GIVES THE n^{TH} MACLAURIN POLYNOMIAL FOR $f(x)$:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

$$= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

THE ERROR $R_n(x)$ MADE IN THE APPROXIMATION $f(x) \approx P_n(x)$ IS BOUNDED BY

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-x_0|^{n+1}$$

WHERE M IS $|f^{(n+1)}(x)| \leq M$ ON THE INTERVAL BETWEEN x_0 AND x .

E.G., THE n^{TH} MACLAURIN POLYNOMIAL FOR $f(x) = \cos x$ IS

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots + (-1)^n \frac{1}{(2n)!}x^{2n}$$

FOR ANY x AND ANY n, $|f^{(n+1)}(x)| \le 1$ SO

$$|R_n(x)| \le \frac{|x|^{n+1}}{(n+1)!}$$

<u>NOTE</u>: FOR ANY FIXED x,

$$\lim_{n \to \infty} \frac{|x|^{n+1}}{(n+1)!} = 0$$

(WE PROVED THIS FOR $x = 10$ SOME TIME AGO AND THE SAME ARGUMENT WORKS FOR ANY x).

WHAT DOES THIS MEAN? AS $n \to \infty$, THE MACLAURIN POLYNOMIALS FOR $\cos x$

$$1$$
$$1 - \frac{1}{2!}x^2$$
$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4$$
$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6$$
$$\vdots$$

CONVERGE TO $\cos x$. BUT THIS IS JUST THE SEQUENCE OF

PARTIAL SUMS OF THE INFINITE SERIES

$$\sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k)!} x^{2k} = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$$

SO WE HAVE

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$$

FOR ANY x.

THE SERIES $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$ IS CALLED THE "MACLAURIN SERIES FOR COS X" AND PROVIDES AN ALTERNATIVE WAY OF THINKING ABOUT A FAMILIAR FUNCTION.

IN GENERAL, THE <u>TAYLOR SERIES FOR</u> $f(x)$ <u>ABOUT</u> $x = x_0$ IS

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots$$

$x_0 = 0$ GIVES THE <u>MACLAURIN SERIES FOR</u> $f(x)$.

SOMETIMES THEY CONVERGE, SOMETIMES THEY DON'T. EVEN WHEN THEY CONVERGE, SOMETIMES THEY CONVERGE TO $f(x)$, SOMETIMES THEY DON'T. WE'LL GET BACK TO QUESTIONS OF CONVERGENCE LATER.

EXAMPLES :

1. $f(x) = e^x$

MACLAURIN SERIES : $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$

$$f(x) = e^x \Rightarrow f^{(k)}(x) = e^x \text{ FOR EVERY } k$$

$$\Rightarrow f^{(k)}(0) = e^0 = 1$$

SO THE MACLAURIN SERIES IS

$$\sum_{k=0}^{\infty} \frac{1}{k!} x^k = 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \dots$$

2. $f(x) = \sin x$ MACLAURIN SERIES :

$f^{(0)}(x) = \sin x$ $f^{(0)}(0) = 0$

$f^{(1)}(x) = \cos x$ $f^{(1)}(0) = 1$

$f^{(2)}(x) = -\sin x$ $f^{(2)}(0) = 0$

$f^{(3)}(x) = -\cos x$ $f^{(3)}(0) = -1$

$f^{(4)}(x) = \sin x$ $f^{(4)}(0) = 0$

⋮ REPEAT ⋮ REPEAT

WRITE OUT THE FIRST FEW NONZERO TERMS OF THE MACLAURIN SERIES AND THE PATTERN WILL BECOME CLEAR :

$$\frac{1}{1!}x^1 - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

AN INFINITE SERIES OF THE FORM

$$\sum_{k=0}^{\infty} c_k (x-x_0)^k = c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + \cdots$$

IS CALLED A <u>POWER SERIES ABOUT</u> $x = x_0$. A <u>POWER SERIES ABOUT</u> $x = 0$ IS ONE OF THE FORM

$$\sum_{k=0}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + \cdots$$

<u>EXAMPLES</u> :

1. THE TAYLOR (MACLAURIN) SERIES FOR ANY FUNCTION $f(x)$.
2. $\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots$ $\quad (c_k = 1)$
3. $\sum_{k=0}^{\infty} \frac{(-1)^k}{3^k(k+1)} x^k$ $\quad \left(c_k = \frac{(-1)^k}{3^k(k+1)}\right)$
4. $\sum_{k=1}^{\infty} \frac{1}{k^2}(x-5)^k$ $\quad \left(c_k = \frac{1}{k^2}\right)$

WE FIND THE VALUES OF X FOR WHICH SUCH SERIES CONVERGE IN THE FOLLOWING WAY :

EXAMPLES :

1. $$\sum_{k=0}^{\infty} \frac{(-1)^k}{3^k(k+1)} x^k$$

APPLY THE RATIO TEST FOR ABSOLUTE CONVERGENCE.

$$\lim_{k\to\infty} \left| \frac{\dfrac{(-1)^{k+1} x^{k+1}}{3^{k+1}(k+2)}}{\dfrac{(-1)^k x^k}{3^k(k+1)}} \right| = \lim_{k\to\infty} \left| \frac{(-1)^{k+1} x^{k+1}}{3^{k+1}(k+2)} \cdot \frac{3^k(k+1)}{(-1)^k x^k} \right|$$

$$= \lim_{k\to\infty} \left| \frac{(-1)\, x\, (k+1)}{3(k+2)} \right|$$

$$= \lim_{k\to\infty} \frac{|x|(k+1)}{3(k+2)}$$

$$= \frac{|x|}{3} \lim_{k\to\infty} \frac{k+1}{k+2}$$

$$= \frac{|x|}{3}$$

SO THE SERIES CONVERGES ABSOLUTELY IF

$$\frac{|x|}{3} < 1$$

$$|x| < 3$$

$$-3 < x < 3$$

AT $x = \pm 3$ THE TEST FAILS SO WE LOOK AT THESE SEPARATELY.

$$x = 3 \ : \quad \sum_{k=0}^{\infty} \frac{(-1)^k}{3^k (k+1)} \cdot 3^k = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k+1}$$

WHICH CONVERGES BY THE ALTERNATING SERIES TEST.

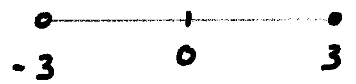
$$x = -3 \ : \quad \sum_{k=0}^{\infty} \frac{(-1)^k}{3^k (k+1)} (-3)^k = \sum_{k=0}^{\infty} \frac{(-1)^k}{3^k (k+1)} (-1)^k \cdot 3^k$$

$$= \sum_{k=0}^{\infty} \frac{1}{k+1}$$

WHICH DIVERGES BY THE INTEGRAL (OR LIMIT COMPARISON) TEST.

THE SERIES CONVERGES FOR x IN THE INTERVAL

$$-3 < x \leq 3$$



2. $$\sum_{k=1}^{\infty} \frac{1}{k^2} (x-5)^k \ :$$

$$\lim_{k \to \infty} \left| \frac{(x-5)^{k+1}}{(k+1)^2} \cdot \frac{k^2}{(x-5)^k} \right| = \lim_{k \to \infty} |x-5| \frac{k^2}{(k+1)^2}$$

$$= |x-5| \lim_{k \to \infty} \frac{k^2}{k^2+2k+1}$$

$$= |x-5|$$

SO THE SERIES CONVERGES ABSOLUTELY WHEN

$$|x-5| < 1$$

$$-1 < x-5 < 1$$

$$4 < x < 6$$

CHECK $x = 4, 6$ (WHERE THE TEST FAILS):

$$x = 4 : \quad \sum_{k=1}^{\infty} \frac{1}{k^2}(4-5)^k = \sum_{k=1}^{\infty} (-1)^k \frac{1}{k^2}$$

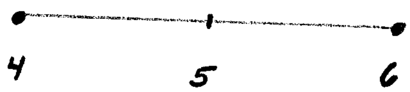
CONVERGES BY THE ALTERNATING SERIES TEST

$$x = 6 : \quad \sum_{k=1}^{\infty} \frac{1}{k^2}(6-5)^k = \sum_{k=1}^{\infty} \frac{1}{k^2}$$

CONVERGES (p-SERIES, $p = 2$)

SERIES CONVERGES FOR

$$4 \le x \le 6$$



WHAT HAPPENED IN THESE TWO EXAMPLES IS TYPICAL:

A POWER SERIES $\sum_{k=0}^{\infty} c_k (x-x_0)^k$ ABOUT x_0

CONVERGES ON SOME SYMMETRIC INTERVAL

ABOUT x_0. THE INTERVAL MAY BE

OPEN : $(x_0-R,\ x_0+R)$ OR $(-\infty,\infty)$

CLOSED : $[x_0-R,\ x_0+R]$ OR $[x_0,x_0]$

HALF-OPEN : $(x_0-R,\ x_0+R]$ OR $[x_0-R,\ x_0+R)$

THIS IS CALLED THE SERIES' INTERVAL OF CONVERGENCE.

R = RADIUS OF CONVERGENCE (FOR $(-\infty,\infty)$, $R=\infty$ AND FOR $[x_0,x_0]$, $R=0$)

E.G., $\sum_{k=0}^{\infty} \frac{(-1)^k}{3^k(k+1)} x^k$ $\quad (-3, 3]$

$R=3$

$\sum_{k=1}^{\infty} \frac{1}{k^2} (x-5)^k$ $\quad [4, 6]$

$R=1$

A (VERY IMPORTANT) NEW POINT-OF-VIEW : MANY FUNCTIONS THAT ARISE IN MATHEMATICS AND ITS APPLICATIONS CANNOT BE EXPRESSED IN TERMS OF THE "ELEMENTARY FUNCTIONS" (POLYNOMIALS, TRIGONOMETRIC, EXPONENTIAL, LOGARITHMIC, HYPERBOLIC, ETC.). THE MOST NATURAL WAY TO VIEW THESE IS AS POWER SERIES. WE WILL SEE EXAMPLES WHEN WE RETURN TO DIFFERENTIAL EQUATIONS.