POLYNOMIAL APPROXIMATION OF FUNCTIONS

SOME FUNCTIONS ARE SIMPLE, E.G.,

$$1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$$

AND SOME ARE NOT, E.G.,

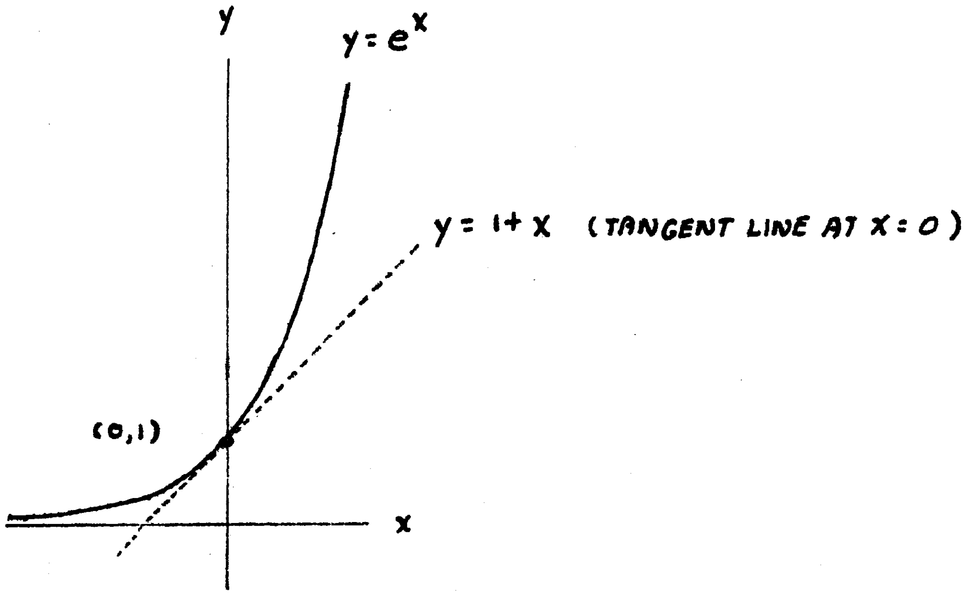
$$e^x$$

(FOR EXAMPLE, WHAT'S THE VALUE OF e^x AT $x = 0.1$?

$$e^{0.1} = e^{\frac{1}{10}} = \sqrt[10]{e} = \sqrt[10]{2.71828\cdots}$$

$$= ? \quad)$$

IT WOULD BE NICE IF COMPLICATED FUNCTIONS COULD BE WELL APPROXIMATED BY SIMPLE FUNCTIONS, E.G.,



AT LEAST NEAR $x = 0$,

$$e^x \approx 1 + x.$$

WITH A LINEAR FUNCTION THIS IS THE BEST YOU CAN DO (TANGENT LINE GOES THROUGH THE SAME POINT $(0,1)$ WITH THE SAME SLOPE 1 AS e^x)

WITH A QUADRATIC FUNCTION YOU CAN DO BETTER (SAME POINT, SAME SLOPE, SAME CONCAVITY)

$$y = a_0 + a_1 x + a_2 x^2$$

$$y(0) = a_0$$

$$y'(0) = a_1$$

$$y''(0) = 2a_2$$

CHOOSE THE COEFFICIENTS SO THAT THESE AGREE WITH e^x, $(e^x)'$ AND $(e^x)''$ AT $x = 0$, I.E.,

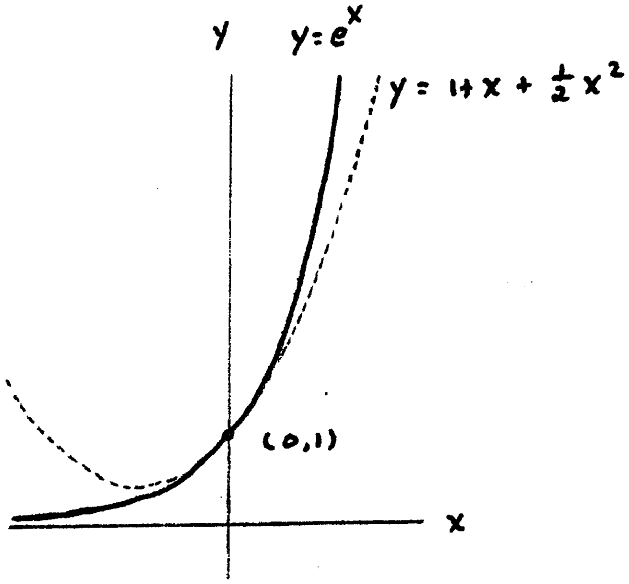
$$a_0 = 1$$

$$a_1 = 1$$

$$2a_2 = 1 \qquad (\text{SO } a_2 = \tfrac{1}{2})$$

$$y = 1 + x + \frac{1}{2}x^2$$

IS THE <u>LOCAL</u> <u>QUADRATIC</u> <u>APPROXIMATION</u> TO e^x AT $x = 0$



WITH HIGHER DEGREE POLYNOMIALS WE CAN FORCE MORE DERIVATIVES AT $x = 0$ TO AGREE WITH THE DERIVATIVES OF e^x AT $x = 0$ AND SO GET A CLOSER FIT.

<u>NOTE</u> : $e^{0.1} \approx 1 + (0.1) + \frac{1}{2}(0.1)^2 = 1 + \frac{1}{10} + \frac{1}{200}$

$$= \frac{221}{200} = 1.105$$

<u>CALCULATOR</u> : $e^{0.1} \approx 1.105170918$

LET'S DO THIS NOW FOR AN ARBITRARY FUNCTION $f(x)$: THE <u>BEST</u> n^{TH} <u>DEGREE</u> <u>POLYNOMIAL</u> <u>APPROXIMATION</u> <u>TO</u> $f(x)$ <u>NEAR</u> $x = 0$.

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_n x^n$$

$P_n(0) = a_0$ CHOOSE $a_0 = f(0)$

$P_n'(0) = a_1$ CHOOSE $a_1 = f'(0)$

$P_n''(0) = 2a_2$ CHOOSE $a_2 = \frac{f''(0)}{2}$

$P_n^{(3)}(0) = 3 \cdot 2 a_3$ CHOOSE $a_3 = \frac{f^{(3)}(0)}{3 \cdot 2}$

$a_4 = \frac{f^{(4)}(0)}{4 \cdot 3 \cdot 2}$

$\vdots$

$a_n = \frac{f^{(n)}(0)}{n!}$

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$ $= n^{\text{TH}}$ <u>MACLAURIN</u> <u>POLYNOMIAL</u> OF $f(x)$

<u>EXAMPLES</u> :

1. FOR $f(x) = e^x$ ALL DERIVATIVES ARE ALSO e^x SO ALL OF THEIR VALUES AT $x = 0$ ARE 1. THUS,

$$P_n(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \dots + \frac{1}{n!}x^n = \sum_{k=0}^{n} \frac{1}{k!} x^k$$

2. COMPUTE $P_7(x)$ FOR $f(x) = \sin x$.

$$f(x) = \sin x \qquad f(0) = 0 \qquad a_0 = 0$$
$$f'(x) = \cos x \qquad f'(0) = 1 \qquad a_1 = 1$$
$$f''(x) = -\sin x \qquad f''(0) = 0 \qquad a_2 = 0$$
$$f^{(3)}(x) = -\cos x \qquad f^{(3)}(0) = -1 \qquad a_3 = -\frac{1}{3!}$$
$$f^{(4)}(x) = \sin x \qquad f^{(4)}(0) = 0 \qquad a_4 = 0$$
$$f^{(5)}(x) = \cos x \qquad f^{(5)}(0) = 1 \qquad a_5 = \frac{1}{5!}$$
$$f^{(6)}(x) = -\sin x \qquad f^{(6)}(0) = 0 \qquad a_6 = 0$$
$$f^{(7)}(x) = -\cos x \qquad f^{(7)}(0) = -1 \qquad a_7 = -\frac{1}{7!}$$

$$P_7(x) = 0 + 1x + 0 - \frac{1}{3!}x^3 + 0 + \frac{1}{5!}x^5 + 0 - \frac{1}{7!}x^7$$
$$= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7$$

<u>NOTE</u>: $\sin 1 \approx P_7(1) = 1 - \frac{1}{6} + \frac{1}{120} - \frac{1}{5040}$

$= 0.841468254$

<u>CALCULATOR</u>: $\sin 1 \approx 0.841470985$

CAN DO SUCH APPROXIMATIONS AT POINTS OTHER THAN $x = 0$ ALSO.
BEST n^{TH} DEGREE POLYNOMIAL APPROXIMATION TO $f(x)$ NEAR $x = x_0$ IS THE <u>TAYLOR</u> <u>POLYNOMIAL</u> OF DEGREE n FOR $f(x)$ AT $x = x_0$ AND IS DEFINED AS FOLLOWS :

$$P_n(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 +$$

$$\frac{f^{(3)}(x_0)}{3!}(x-x_0)^3 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \qquad (x_0 = 0 \text{ GIVES MACLAURIN })$$

<u>EXAMPLE</u> : FIND THE TAYLOR POLYNOMIAL OF DEGREE 5 FOR $f(x) = \cos x$ AT $x = \frac{\pi}{2}$.

$$f(x) = \cos x \qquad f(\tfrac{\pi}{2}) = 0$$

$$f'(x) = -\sin x \qquad f'(\tfrac{\pi}{2}) = -1$$

$$f''(x) = -\cos x \qquad f''(\tfrac{\pi}{2}) = 0$$

$$f^{(3)}(x) = \sin x \qquad f^{(3)}(\tfrac{\pi}{2}) = 1$$

$$f^{(4)}(x) = \cos x \qquad f^{(4)}(\tfrac{\pi}{2}) = 0$$

$$f^{(5)}(x) = -\sin x \qquad f^{(5)}(\tfrac{\pi}{2}) = -1$$

$$P_5(x) = 0 + (-1)(x-\tfrac{\pi}{2}) + 0 + \frac{1}{3!}(x-\tfrac{\pi}{2})^3 + 0 + \frac{-1}{5!}(x-\tfrac{\pi}{2})^5$$

$$= -(x-\tfrac{\pi}{2}) + \frac{1}{3!}(x-\tfrac{\pi}{2})^3 - \frac{1}{5!}(x-\tfrac{\pi}{2})^5$$

IT'S POSSIBLE TO MEASURE THE ACCURACY OF THESE POLYNOMIAL APPROXIMATIONS :

GIVEN $f(x)$, x_0 AND n, LET

$$R_n(x) = f(x) - P_n(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{n!}(x-x_0)$$

= <u>REMAINDER</u> (<u>ERROR</u>) MADE IN APPROXIMATING $f(x)$ BY $P_n(x)$.

<u>THEOREM</u> : IF, ON SOME INTERVAL I CONTAINING x_0 ,

$$|f^{(n+1)}(x)| \leq M$$

FOR SOME CONSTANT M, THEN

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-x_0|^{n+1}$$

FOR ALL x IN I.

<u>EXAMPLES</u> :

1. EARLIER WE USED THE 7^{TH} DEGREE MACLAURIN POLYNOMIAL TO APPROXIMATE SIN 1.

$f(x) = \sin x$ $\qquad$ $x_0 = 0$

$n = 7$ $\qquad$ $x = 1$

ON THE INTERVAL $I = [0,1]$ (OR ANY OTHER INTERVAL)

$$|f^{(8)}(x)| \le 1$$

THUS,

$$|R_7(1)| \le \frac{1}{8!}(1-0)^8 = \frac{1}{8!} = \frac{1}{40320} = 0.000024802$$

2. EARIER WE USED THE 2^{ND} DEGREE MACLAURIN POLYNOMIAL $P_2(x)$ FOR $f(x) = e^x$ TO APPROXIMATE $e^{0.1}$.

<u>HOW LARGE MUST n BE CHOSEN IN ORDER FOR $P_n(0.1)$ TO APPROXIMATE $e^{0.1}$ TO FIVE DECIMAL PLACE ACCURACY</u>, I.E., SO THAT

$$|R_n(0.1)| < 0.000005$$

$f(x) = e^x \Rightarrow f^{(n+1)}(x) = e^x$ AND SO, ON THE INTERVAL $I = [0, 0.1]$,

$$|f^{n+1}(x)| < 2$$

[$f^{(n+1)}(x) = e^x$ IS INCREASING ON $[0, 0.1]$ SO ITS LARGEST VALUE IS $e^{0.1}$ WHICH IS SURELY LESS THAN 2 BECAUSE $2^{10} > e$] THUS,

$$|R_n(0.1)| < \frac{2}{(n+1)!}(0.1-0)^{n+1} = \frac{2}{(n+1)!}\left(\frac{1}{10}\right)^{n+1}$$

THUS, WE WANT TO CHOOSE n SO THAT

$$\frac{2}{10^{n+1}(n+1)!} < 0.000005$$

$$\frac{1}{10^{n+1}(n+1)!} < 0.0000025$$

$$10^{n+1}(n+1)! > \frac{1}{0.0000025}$$

$$10^{n+1}(n+1)! > 400{,}000$$

CHECK A FEW VALUES OF n :

$n=2$: $10^3(3!) = 6000$

$n=3$: $10^4(4!) = 240{,}000$

$n=4$: $10^5(5!) = 12{,}000{,}000$

SO $n=4$ WILL DO.

IT WOULD BE PARTICULARLY NICE IF $|R_n(x)| \longrightarrow 0$ AS $n \to \infty$ FOR THEN WE COULD GET ARBITRARILY GOOD APPROXIMATIONS JUST BY CHOOSING THE DEGREE HIGH ENOUGH.

WE RETURN TO THIS IN THE NEXT SECTION.