PARTIAL FRACTIONS

SOME INTEGRALS (OF RATIONAL FUNCTIONS) THAT WE ALREADY KNOW HOW TO DO :

$$\int \frac{1}{2x+3}\,dx = \frac{1}{2}\ln|2x+3| + C$$

$$\int \frac{1}{(2x+3)^5}\,dx = -\frac{1}{8}\frac{1}{(2x+3)^4} + C$$

$$\int \frac{1}{x^2-6x+9}\,dx = \int \frac{1}{(x-3)^2}\,dx = -\frac{1}{x-3} + C$$

$$\int \frac{1}{4x^2+9}\,dx = \frac{1}{6}\arctan\left(\frac{2x}{3}\right) + C$$

$$\int \frac{x}{(4x^2+9)^2}\,dx = -\frac{1}{8}\frac{1}{4x^2+9} + C$$

$$\int \frac{1}{x^2-4x+5}\,dx = \int \frac{1}{(x-2)^2+1}\,dx = \arctan(x-2) + C$$

$$\int \frac{x}{x^2-4x+8}\,dx = \int \frac{x}{(x-2)^2+4}\,dx$$

$$= \int \frac{(x-2)+2}{(x-2)^2+4}\,dx$$

$$= \frac{1}{2}\ln(x^2-4x+8) + \arctan\left(\frac{x-2}{2}\right) + C$$

THERE IS A THEOREM IN ALGEBRA ("PARTIAL FRACTIONS DECOMPOSITION") THAT SAYS ESSENTIALLY THAT THE INTEGRAL OF ANY RATIONAL FUNCTION CAN BE WRITTEN AS A SUM OF INTEGRALS OF THESE TYPES.

FIRST NOTE THAT, BY FIRST PERFORMING A LONG DIVISION, WE CAN ASSUME THAT THE DEGREE OF THE NUMERATOR IS LESS THAN THE DEGREE OF THE DENOMINATOR, E.G., FOR

$$\frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}$$

WE DIVIDE

$$\begin{array}{r l} & 3x^2 + 1 \\ x^2+x-2 \,\big) & \overline{3x^4+3x^3-5x^2+x-1} \\ & \underline{3x^4+3x^3-6x^2} \\ & \qquad\qquad x^2+x-1 \\ & \qquad\qquad \underline{x^2+x-2} \\ & \qquad\qquad\qquad 1 \end{array}$$

TO GET

$$\frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2} = 3x^2+1 + \frac{1}{x^2+x-2}$$

THUS, IF WE NEEDED TO INTEGRATE THIS THE POLYNOMIAL PART WOULD BE EASY AND WE NEED ONLY WORRY ABOUT THE REMAINDER TERM.

"PARTIAL FRACTIONS" IS ESSENTIALLY THE REVERSE OF ADDING FRACTIONS WITH DIFFERENT DENOMINATORS, E.G., EVERYONE KNOWS HOW TO DO THIS

$$\frac{2}{x-4} + \frac{3}{x+1} = \frac{2}{x-4}\,\frac{x+1}{x+1} + \frac{3}{x+1}\,\frac{x-4}{x-4}$$

$$= \frac{2(x+1) + 3(x-4)}{(x-4)(x+1)}$$

$$= \frac{5x-10}{(x-4)(x+1)}$$

$$= \frac{5x-10}{x^2-3x-4}$$

BUT WE WANT TO GO BACKWARDS, I.E., GIVEN $\frac{5x-10}{x^2-3x-4}$, WE WANT TO FIND

$$\frac{5x-10}{x^2-3x-4} = \frac{2}{x-4} + \frac{3}{x+1}$$

(ITS "PARTIAL FRACTIONS DECOMPOSITION").

THE PROCEDURE CAN BE OUTLINED IN A NUMBER OF STEPS WHICH WE DESCRIBE AS FOLLOWS :

> <u>NOTE</u> : ALL OF THE FOLLOWING REQUIRES THAT THE DEGREE OF THE NUMERATOR IS <u>LESS THAN</u> THE DEGREE OF THE DENOMINATOR.

TO FIND THE PARTIAL FRACTIONS DECOMPOSITION OF THE RATIONAL FUNCTION

$$\frac{P(x)}{Q(x)}$$

WHERE degree $(P(x)) <$ degree $(Q(x))$:

FIRST, FACTOR THE DENOMINATOR $Q(x)$ COMPLETELY (I.E., INTO A PRODUCT OF LINEAR AND IRREDUCIBLE QUADRATIC FACTORS) AND WRITE ANY REPEATED FACTORS AS POWERS.

THE REMAINING STEPS DEPEND ON WHAT KIND OF FACTORS THE DENOMINATOR HAS AND WE WILL ILLUSTRATE ALL OF THE POSSIBILITIES WITH EXAMPLES.

1. <u>DISTINCT</u> <u>LINEAR</u> <u>FACTORS</u> (<u>NOT</u> <u>REPEATED</u>) :

THE PARTIAL FRACTIONS DECOMPOSITION CONTAINS A TERM WITH A CONSTANT OVER EACH DISTINCT LINEAR FACTOR, E.G.,

$$\frac{5x-10}{x^2-3x-4} = \frac{5x-10}{(x-4)(x+1)} = \frac{A}{x-4} + \frac{B}{x+1}$$

TO FIND A AND B, CLEAR FRACTIONS

$$\left[\frac{5x-10}{(x-4)(x+1)} = \frac{A}{x-4} + \frac{B}{x+1} \right](x-4)(x+1)$$

$$5x-10 = A(x+1) + B(x-4)$$

IN THIS CASE THERE ARE TWO POSSIBLE WAYS TO PROCEED.

(i) $5x - 10 = (A+B)x + (A-4B)$ IMPLIES

$$\begin{cases} A + B = 5 \\ A - 4B = -10 \end{cases}$$

NOW SOLVE THE SYSTEM OF EQUATIONS, E.G.,

$$\begin{cases} A + B = 5 \\ -A + 4B = 10 \end{cases}$$

$$5B = 15$$

$$B = 3$$

$$A = 5 - B = 5 - 3 = 2$$

THUS, $A = 2$, $B = 3$ SO

$$\frac{5x-10}{x^2-3x-4} = \frac{2}{x-4} + \frac{3}{x+1}$$

(ii) INTO

$$5x-10 = A(x+1) + B(x-4)$$

SUBSTITUTE FIRST $x = 4$ TO OBTAIN

$$20 - 10 = A(5) + B(0)$$
$$10 = 5A$$
$$2 = A$$

AND THEN $x = -1$ TO OBTAIN

$$-5 - 10 = A(0) + B(-5)$$
$$-15 = -5B$$
$$3 = B$$

HOWEVER YOU DO IT WE CAN NOW INTEGRATE $\dfrac{5x-10}{x^2-3x-4}$:

$$\int \frac{5x-10}{x^2-3x-4}\,dx = \int \left(\frac{2}{x-4} + \frac{3}{x+1}\right)dx$$
$$= 2\ln|x-4| + 3\ln|x+1| + C$$

<u>ANOTHER</u> <u>EXAMPLE</u> : $\displaystyle\int \frac{dx}{x(x^2-1)}$

$$\frac{1}{x(x^2-1)} = \frac{1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$
$$1 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$$
$$= Ax^2 - A + Bx^2 + Bx + Cx^2 - Cx$$
$$0x^2 + 0x + 1 = (A+B+C)x^2 + (B-C)x - A$$

SO WE MUST HAVE

$$\begin{cases} A + B + C = 0 \\ \quad\; B - C = 0 \\ -A = 1 \end{cases}$$

THUS, $A = 1$ SO

$$\begin{cases} 1 + B + C = 0 \\ \quad\; B - C = 0 \end{cases}$$

$$1 + 2B = 0$$

$$B = -\frac{1}{2}$$

$$C = B = -\frac{1}{2}$$

CONCLUSION :

$$\frac{1}{x(x^2-1)} = \frac{1}{x} + \frac{-\frac{1}{2}}{x-1} + \frac{-\frac{1}{2}}{x+1}$$

$$\int \frac{dx}{x(x^2-1)} = \int \left(\frac{1}{x} - \frac{1}{2}\,\frac{1}{x-1} - \frac{1}{2}\,\frac{1}{x+1} \right) dx$$

$$= \ln|x| - \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C$$

2. <u>REPEATED LINEAR FACTORS</u> :

CONSTANT OVER EVERY POWER OF THE REPEATED LINEAR FACTOR UP TO AND INCLUDING THE POWER THAT APPEARS, E.G.,

$$\frac{x^2}{x^3+3x^2+3x+1} = \frac{x^2}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

$$x^2 = A(x+1)^2 + B(x+1) + C$$

$$= A(x^2+2x+1) + Bx + B + C$$

$$1x^2 + 0x + 0 = Ax^2 + (2A+B)x + (A+B+C)$$

$$\begin{cases} A = 1 \\ 2A + B = 0 \\ A + B + C = 0 \end{cases}$$

$$A = 1$$

$$B = -2A = -2$$

$$C = -A - B = -1 - (-2) = 1$$

SO

$$\frac{x^2}{x^3+3x^2+3x+1} = \frac{1}{x+1} - \frac{2}{(x+1)^2} + \frac{1}{(x+1)^3}$$

THUS, WE CAN INTEGRATE :

$$\int \frac{x^2}{x^3+3x^2+3x+1}\,dx = \int \left(\frac{1}{x+1} - 2(x+1)^{-2} + (x+1)^{-3} \right) dx$$

$$= \ln|x+1| + \frac{2}{x+1} - \frac{1}{2(x+1)^2} + C$$

NOTE : BOTH TYPES CAN OCCUR, E.G., THE PARTIAL FRACTIONS DECOMPOSITION FOR

$$\frac{2x^2+3}{x^3-2x^2+x} = \frac{2x^2+3}{x(x^2-2x+1)} = \frac{2x^2+3}{x(x-1)^2}$$

LOOKS LIKE

$$\frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

3. IRREDUCIBLE QUADRATIC FACTORS (NOT REPEATED):

THE PARTIAL FRACTIONS DECOMPOSITION CONTAINS A TERM WITH A LINEAR POLYNOMIAL (E.G., $Ax+B$) OVER EACH DISTINCT IRREDUCIBLE QUADRATIC FACTOR, E.G.,

$$\frac{x^2+x-2}{3x^3-x^2+3x-1} = \frac{x^2+x-2}{x^2(3x-1)+(3x-1)}$$

$$= \frac{x^2+x-2}{(3x-1)(x^2+1)} = \frac{A}{3x-1} + \frac{Bx+C}{x^2+1}$$

$$x^2+x-2 = A(x^2+1)+(Bx+C)(3x-1)$$

$$= Ax^2+A+3Bx^2-Bx+3Cx-C$$

$$1x^2+1x+(-2) = (A+3B)x^2+(-B+3C)x+(A-C)$$

SO

$$\begin{cases} A + 3B & = 1 \\ \quad -B + 3C & = 1 \\ A \quad - C & = -2 \end{cases}$$

THE FIRST TWO GIVE

$$\begin{cases} A + 3B & = 1 \\ \quad -3B + 9C & = 3 \end{cases}$$
$$\overline{A \quad + 9C \quad = 4}$$

COMBINING THIS WITH THE THIRD GIVES

$$\begin{cases} A + 9C & = 4 \\ A - C & = -2 \end{cases}$$
$$\overline{\quad 10C = 6 \quad}$$

$$C = \frac{3}{5}$$

$$A = C - 2 = \frac{3}{5} - 2 = -\frac{7}{5}$$

$$B = 3C - 1 = \frac{9}{5} - 1 = \frac{4}{5}$$

SO

$$\frac{x^2+x-2}{3x^3-x^2+3x-1} = \frac{-\frac{7}{5}}{3x-1} + \frac{\frac{4}{5}x + \frac{3}{5}}{x^2+1}$$

$$= -\frac{7}{5}\,\frac{1}{3x-1} + \frac{1}{5}\left(\frac{4x+3}{x^2+1}\right)$$

AND WE CAN INTEGRATE

$$\int \frac{x^2+x-2}{3x^3-x^2+3x-1}\,dx = -\frac{7}{5}\int \frac{1}{3x-1}\,dx + \frac{1}{5}\int \frac{4x}{x^2+1}\,dx + \frac{1}{5}\int \frac{3}{x^2+1}\,dx$$

$$= -\frac{7}{5}\frac{1}{3}\int \frac{1}{3x-1}(3\,dx) + \frac{2}{5}\int \frac{1}{x^2+1}(2x\,dx)$$

$$+ \frac{3}{5}\int \frac{1}{x^2+1}\,dx$$

$$= -\frac{7}{15}\ln|3x-1| + \frac{2}{5}\ln(x^2+1)$$

$$+ \frac{3}{5}\text{ARCTAN } x + C$$

4. <u>REPEATED IRREDUCIBLE QUADRATIC FACTORS</u> :

LINEAR POLYNOMIAL OVER EVERY POWER UP TO AND INCLUDING THE ONE THAT APPEARS, E.G.,

$$\frac{3x^4+4x^3+16x^2+20x+9}{(x+2)(x^2+3)^2} = \frac{A}{x+2} + \frac{Bx+C}{x^2+3} + \frac{Dx+E}{(x^2+3)^2}$$

$$3x^4+4x^3+16x^2+20x+9 = A(x^2+3)^2 + (Bx+C)(x+2)(x^2+3)$$

$$+ (Dx+E)(x+2)$$

$$= (A+B)x^4 + (2B+C)x^3 +$$

$$(6A+3B+2C+D)x^2 +$$

$$(6B+3C+2D+E)x +$$

$$(9A+6C+2E)$$

EQUATING COEFFICIENTS AT THIS POINT YIELDS A HUGE AND RATHER NASTY SYSTEM OF EQUATIONS TO SOLVE FOR A, B, C, D, E.

IT HELPS A LOT TO NOTICE THAT WE CAN AT LEAST DETERMINE A BY SUBSTITUTING $x = -2$ INTO

$$3x^4+4x^3+16x^2+20x+9 = A(x^2+3)^2+(Bx+C)(x+2)(x^2+3) + (Dx+E)(x+2)$$

THE RESULT IS

$$A = 1$$

USE THIS VALUE OF A AND EQUATE COEFFICIENTS :

$$\left\{\begin{array}{lcl} 1+B & = & 3 \\ 2B+C & = & 4 \\ 6+3B+2C+D & = & 16 \\ 6B+3C+2D+E & = & 20 \\ 9+6C+2E & = & 9 \end{array}\right.$$

THUS,

$$\begin{aligned} B &= 2 \\ C &= 4-2B = 4-4 = 0 \\ D &= 16-6-3B-2C = 4 \\ E &= 20-6B-3C-2D = 0 \end{aligned}$$

(THE LAST EQUATION IS AUTOMATICALLY SATISFIED WITH $C = E = 0$).

THUS,

$$\frac{3x^4+4x^3+16x^2+20x+9}{(x+2)(x^2+3)^2} = \frac{1}{x+2} + \frac{2x}{x^2+3} + \frac{4x}{(x^2+3)^2}$$

SO $$\int \frac{3x^4+4x^3+16x^2+20x+9}{(x+2)(x^2+3)^2}\,dx = \int \frac{1}{x+2}\,dx + \int \frac{1}{x^2+3}(2x\,dx)$$

$$+ 2\int (x^2+3)^{-2}(2x\,dx)$$

$$= \ln|x+2| + \ln(x^2+3)$$

$$- \frac{2}{x^2+3} + C$$