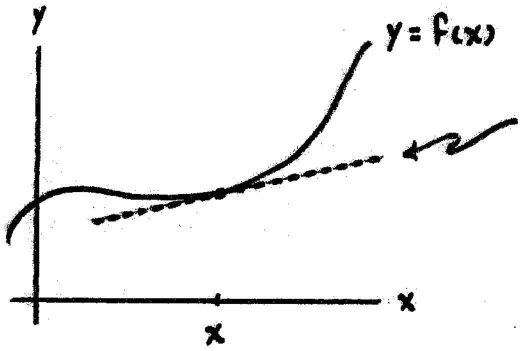
PARTIAL DERIVATIVES

RECALL : GIVEN $y = f(x)$ AND x IN ITS DOMAIN.



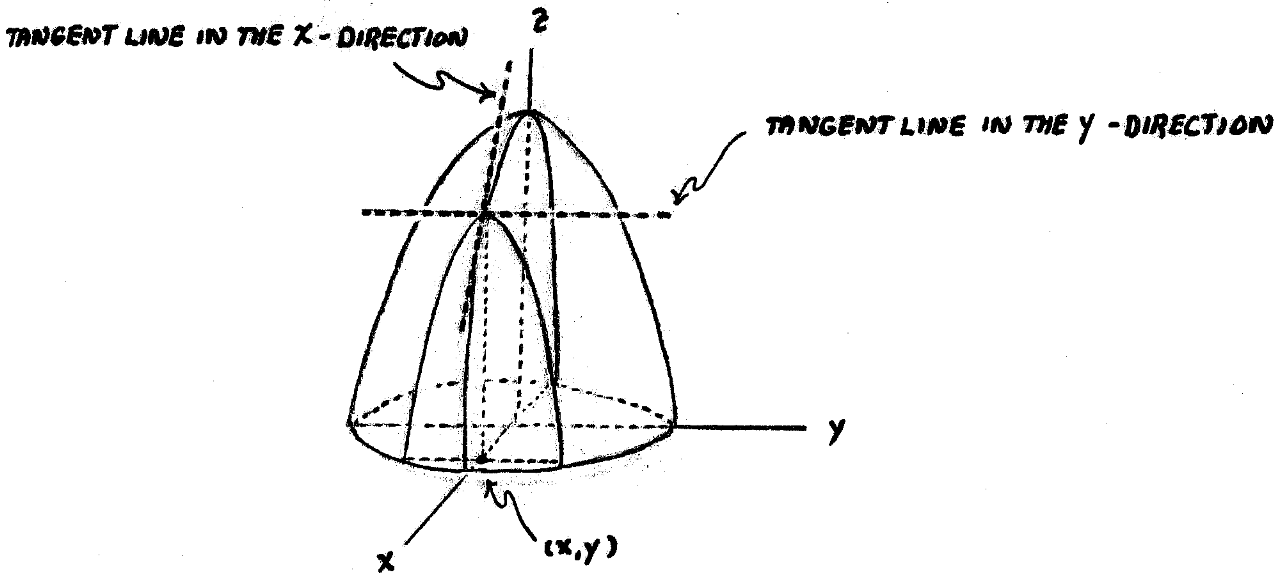
$$\text{SLOPE} = f'(x) = \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

= MEASURE OF THE RATE AT WHICH f IS CHANGING AT x

NOW SUPPOSE $z = f(x,y)$ AND (x,y) IS A POINT IN ITS DOMAIN.

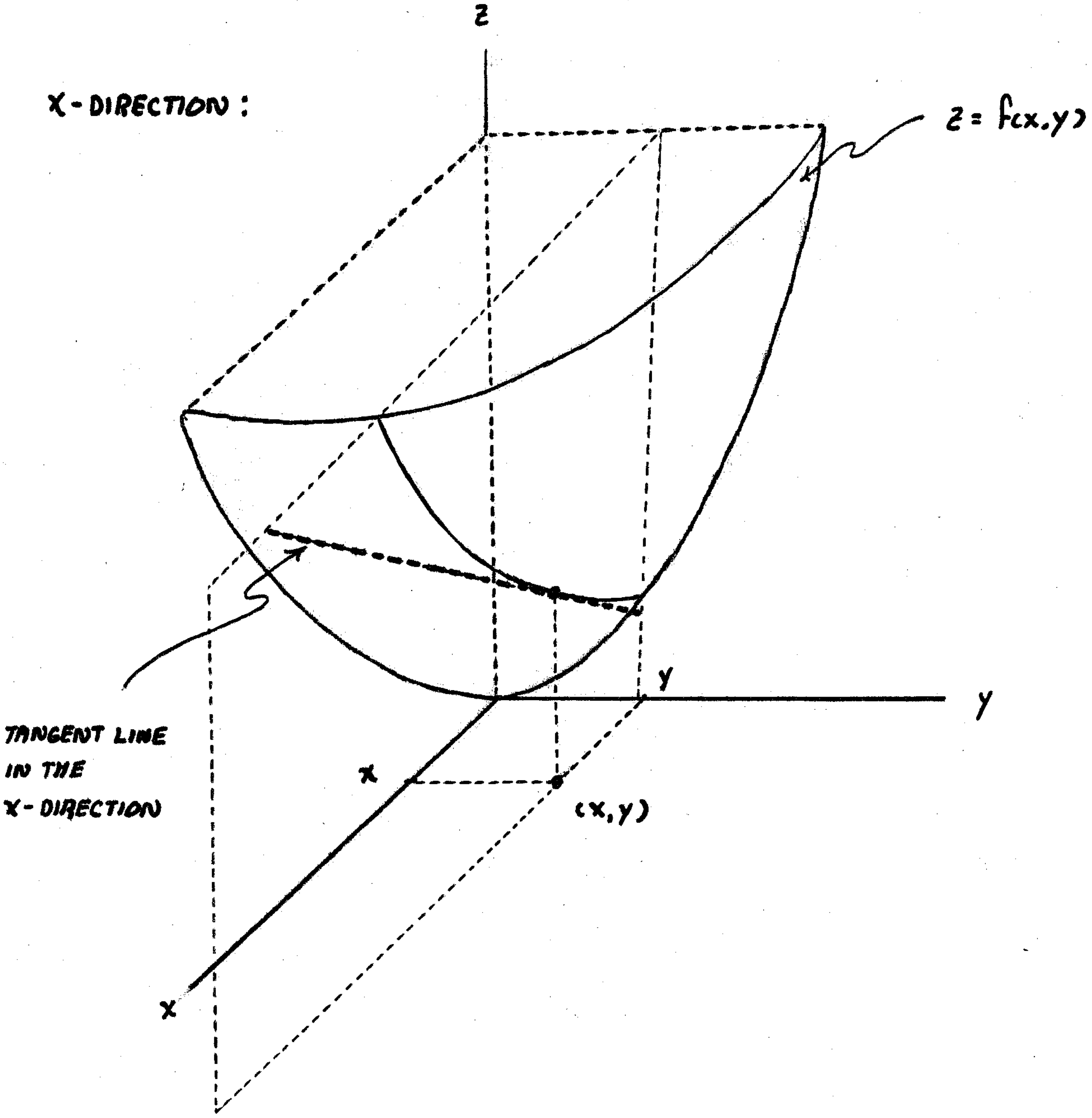
" RATE AT WHICH f IS CHANGING AT (x,y) "

MAKES NO SENSE SINCE f CAN CHANGE AT DIFFERENT RATES IN DIFFERENT DIRECTIONS AT (x,y) , E.G.,



WE WILL BEGIN BY COMPUTING RATES OF CHANGE IN THE X-DIRECTION AND IN THE Y-DIRECTION.

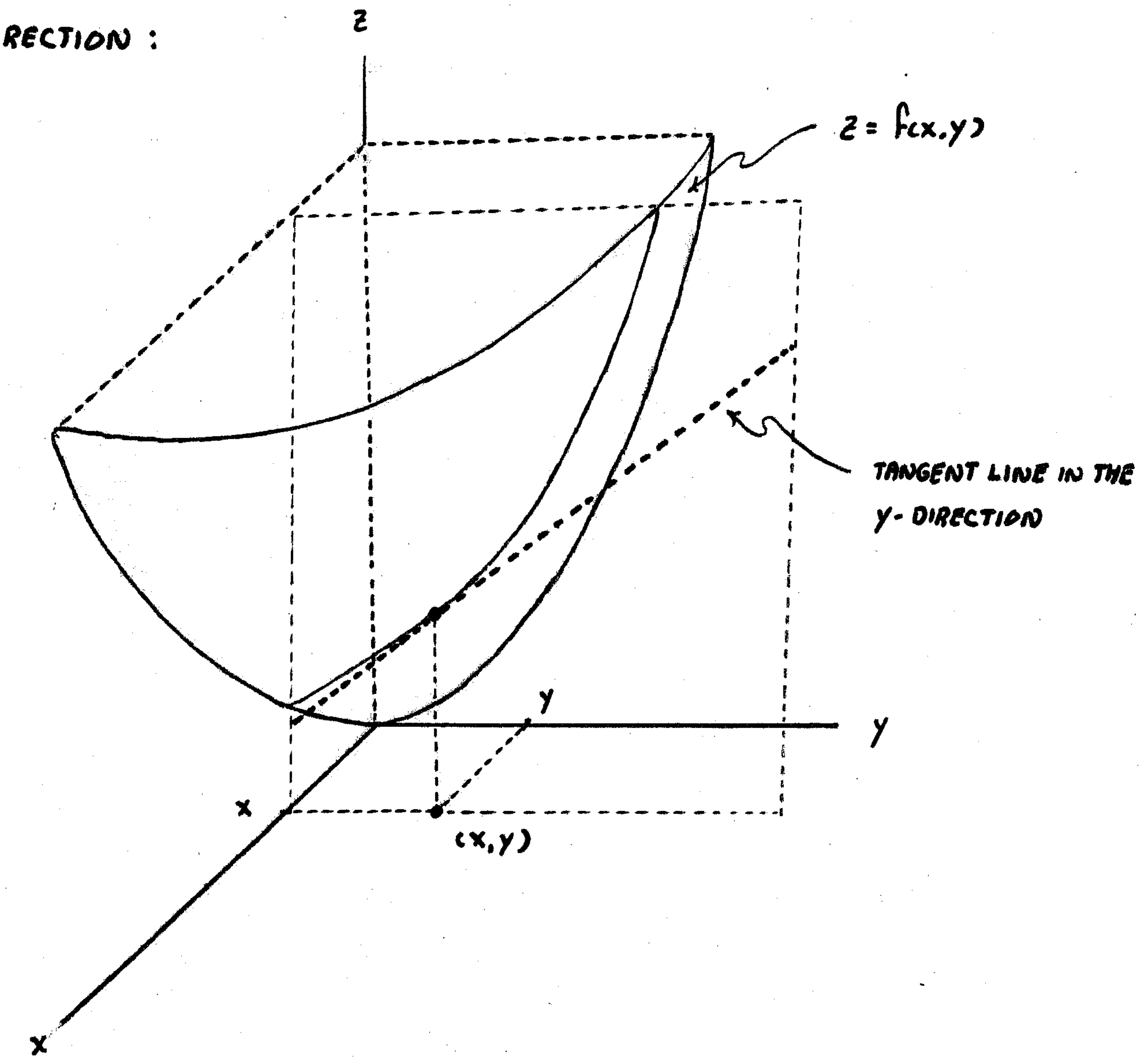
X-DIRECTION :



SLOPE OF TANGENT LINE IN X-DIRECTION = <u>PARTIAL DERIVATIVE OF</u> f <u>WITH RESPECT TO</u> x

$$= \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

(HOLD y FIXED AND DIFFERENTIATE WITH RESPECT TO x AS USUAL) y - DIRECTION :



SLOPE OF TANGENT LINE IN y- DIRECTION = <u>PARTIAL</u> <u>DERIVATIVE</u> <u>OF</u> f <u>WITH</u> <u>RESPECT</u> <u>TO</u> y

$$= \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h}$$

(HOLD x FIXED AND DIFFERENTIATE WITH RESPECT TO y AS USUAL)

SOME ALTERNATE NOTATIONS : $z = f(x,y)$

$$\frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(f(x,y)) = z_x = D_x f = D_1 f = \cdots$$

$$\frac{\partial f}{\partial y} = \frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(f(x,y)) = z_y = D_y f = D_2 f = \cdots$$

<u>EXAMPLES</u> :

1. $z = f(x,y) = 2x + 5x^4y^3$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(2x + 5x^4y^3) = 2 + (20x^3)y^3 = 2 + 20x^3y^3$$

$$z_y = \frac{\partial}{\partial y}(2x + 5x^4y^3) = 0 + (5x^4)(3y^2) = 15x^4y^2$$

$$\frac{\partial f}{\partial x}(1,2) = 2 + 20(1^3)(2^3) = 162$$

$$z_y(1,2) = 15(1^4)(2^2) = 60$$

2. $z = f(x,y) = (x^2+y^2)e^{x^2y}$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}((x^2+y^2)e^{x^2y})$$

$$= (x^2+y^2)\frac{\partial}{\partial x}(e^{x^2y}) + e^{x^2y}\frac{\partial}{\partial x}(x^2+y^2)$$

$$= (x^2+y^2)e^{x^2y}\frac{\partial}{\partial x}(x^2y) + e^{x^2y}(2x+0)$$

$$= 2xy(x^2+y^2)e^{x^2y} + 2xe^{x^2y}$$

3. $R(s,t) = s^4 \sin(st^3)$

$$\frac{\partial R}{\partial s} = \frac{\partial}{\partial s}(s^4 \sin(st^3))$$

$$= s^4 \frac{\partial}{\partial s}(\sin(st^3)) + \sin(st^3)\frac{\partial}{\partial s}(s^4)$$

$$= s^4 \cos(st^3)\frac{\partial}{\partial s}(st^3) + 4s^3 \sin(st^3)$$

$$= s^4 t^3 \cos(st^3) + 4s^3 \sin(st^3)$$

4. FIND $\frac{\partial z}{\partial x}$ BY IMPLICIT DIFFERENTIATION IF

$$x \sin(y+z) - xy + z^2 = 0.$$

(z IS THE FUNCTION, x THE INDEPENDENT VARIABLE AND y IS HELD FIXED)

$$\frac{\partial}{\partial x}(x \sin(y+z) - xy + z^2) = \frac{\partial}{\partial x}(0)$$

$$\frac{\partial}{\partial x}(x \sin(y+z)) - \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial x}(z^2) = 0$$

$$x \frac{\partial}{\partial x}\sin(y+z) + \sin(y+z)\frac{\partial}{\partial x}(x) - y + 2z\frac{\partial z}{\partial x} = 0$$

$$x \cos(y+z)\frac{\partial}{\partial x}(y+z) + \sin(y+z) - y + 2z\frac{\partial z}{\partial x} = 0$$

$$x \cos(y+z)(0 + \frac{\partial z}{\partial x}) + \sin(y+z) - y + 2z\frac{\partial z}{\partial x} = 0$$

$$[x \cos(y+z) + 2z]\frac{\partial z}{\partial x} = y - \sin(y+z)$$

$$\frac{\partial z}{\partial x} = \frac{y - \sin(y+z)}{x \cos(y+z) + 2z}$$

SECOND ORDER DERIVATIVES (A NOTATIONAL ANNOYANCE):

$$\begin{tikzcd}[column sep=small]
 & & z=f(x,y) \arrow[dl] \arrow[dr] & & \\
 & \frac{\partial z}{\partial x} \arrow[dl] \arrow[d] & & \frac{\partial z}{\partial y} \arrow[d] \arrow[dr] & \\
\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x^2} & \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial y\partial x} & & \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial x\partial y} & \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial y^2}
\end{tikzcd}$$

MIXED SECOND ORDER
PARTIAL DERIVATIVES

$$\begin{tikzcd}[column sep=small]
 & & z=f(x,y) \arrow[dl] \arrow[dr] & & \\
 & z_x \arrow[dl] \arrow[d] & & z_y \arrow[d] \arrow[dr] & \\
(z_x)_x = z_{xx} & (z_x)_y = z_{xy} & & (z_y)_x = z_{yx} & (z_y)_y = z_{yy}
\end{tikzcd}$$

NOTE THAT $z_{xy} = \dfrac{\partial^2 z}{\partial y \partial x}$

AND $z_{yx} = \dfrac{\partial^2 z}{\partial x \partial y}$

EXAMPLE : $z = f(x,y) = 2x + 5x^4y^3$

$$z_x = 2 + 20x^3y^3 \quad : \quad z_{xx} = 60x^2y^3 \qquad z_{xy} = 60x^3y^2$$

$$z_y = 15x^4y^2 \quad : \quad z_{yy} = 30x^4y \qquad z_{yx} = 60x^3y^2$$

NOTE : $z_{xy} = z_{yx}$

THIS IS NO ACCIDENT :

THEOREM : LET $z = f(x,y)$. IF z_{xy} AND z_{yx} ARE BOTH CONTINUOUS, THEN

$$z_{xy} = z_{yx}$$

(I.E.,

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y} \quad)$$

ALL OF THIS EXTENDS EASILY TO FUNCTIONS OF MORE THAN TWO VARIABLES, E.G.,

$w = z \text{ ARCTAN}\left(\frac{y}{x}\right)$:

$$\frac{\partial w}{\partial x} = z \frac{\partial}{\partial x} \text{ARCTAN}\left(\frac{y}{x}\right)$$

$$= z \left(\frac{1}{1 + \left(\frac{y}{x}\right)^2} \right) \frac{\partial}{\partial x}\left(\frac{y}{x}\right)$$

$$= z \left(\frac{1}{1 + \frac{y^2}{x^2}} \right) \left(- \frac{y}{x^2} \right) = - \frac{yz}{x^2 + y^2}$$

$$\frac{\partial w}{\partial z} = \frac{\partial}{\partial z}\left(z \text{ ARCTAN}\left(\frac{y}{x}\right)\right) = \text{ARCTAN}\left(\frac{y}{x}\right)$$

ONE MORE EXAMPLE : SHOW THAT $u(x,y) = \ln\sqrt{x^2+y^2}$ SATISFIES

$$u_{xx} + u_{yy} = 0$$

(THE " 2-DIMENSIONAL LAPLACE EQUATION ").

$$u = \ln\sqrt{x^2+y^2} = \frac{1}{2}\ln(x^2+y^2) \Rightarrow$$

$$u_x = \frac{1}{2}\frac{1}{x^2+y^2}(2x) = \frac{x}{x^2+y^2}$$

$$u_{xx} = \frac{(x^2+y^2)\frac{\partial}{\partial x}(x) - x\frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)^2}$$

$$= \frac{(x^2+y^2) - 2x^2}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

SIMILARLY,

$$u_{yy} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

SO

$$u_{xx}+u_{yy} = \frac{-x^2+y^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$= \frac{-x^2+y^2+x^2-y^2}{(x^2+y^2)^2}$$

$$= \frac{0}{(x^2+y^2)^2} = 0$$