NEWTON'S METHOD

OUR DISCUSSION OF "NEWTON'S METHOD" WILL BE QUITE BRIEF SINCE IT IS THE IDEA BEHIND THE METHOD RATHER THAN DETAILED APPLICATIONS THAT INTEREST US HERE.

NOTE: SOME EQUATIONS ARE EASY TO SOLVE EXPLICITLY, E.G.,

$$2x + 7 = 0$$

$$x^2 + 4x + 2 = 0$$

$$x^3 + 2x^2 - 3x = 0$$

SOME ARE NOT, E.G.,

$$x^3 - 2x - 5 = 0$$

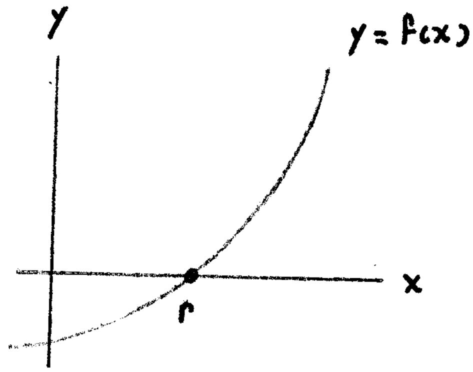
$$\cos x - x = 0$$

FOR THE SECOND TYPE ONE MUST OFTEN BE CONTENT WITH APPROXIMATE SOLUTIONS.

NEWTON DEVISED A SIMPLE PROCEDURE WHICH OFTEN PRODUCES A SEQUENCE OF BETTER AND BETTER APPROXIMATIONS TO SOLUTIONS TO EQUATIONS LIKE

$$f(x) = 0 .$$

GEOMETRICALLY, WE WANT TO APPROXIMATE THE "r" SHOWN BELOW:



I WILL ILLUSTRATE THE IDEA FOR THE EQUATION $x^3 - 2x - 5 = 0$ AND THEN GIVE AN ALGORITHM FOR OBTAINING THE APPROXIMATIONS WITH ALL OF THE GEOMETRY PRUNED AWAY.

THUS, WE LET

$$f(x) = x^3 - 2x - 5$$

AND APPROXIMATE SOLUTIONS TO

$$f(x) = 0.$$

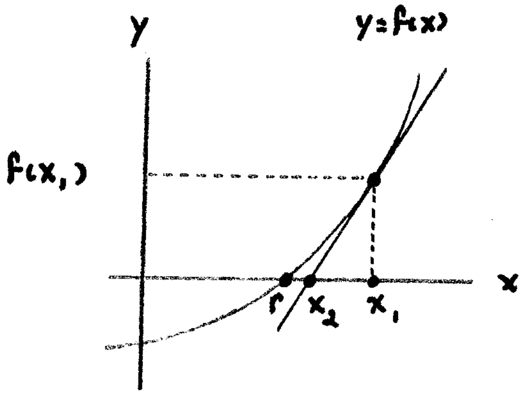
<u>PROCEDURE</u> :

1. "GUESS" A FIRST APPROXIMATION x_1 TO r.

   <u>EXAMPLE</u> : FOR $f(x) = x^3 - 2x - 5$ WE NOTICE THAT $f(2) = -1$ AND $f(3) = 16$ SO THE INTERMEDIATE VALUE THEOREM SAYS THAT THERE MUST BE A SOLUTION BETWEEN 2 AND 3. THUS, WE MIGHT TAKE

$$x_1 = 2$$

2. THE PICTURE



SUGGESTS THAT WE TAKE AS OUR SECOND APPROXIMATION x_2 THE x-INTERCEPT OF THE TANGENT LINE TO THE GRAPH OF $y = f(x)$ AT x_1.

TANGENT LINE : $y - f(x_1) = f'(x_1)(x - x_1)$

x-INTERCEPT : $0 - f(x_1) = f'(x_1)(x_2 - x_1)$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

(PROVIDED $f'(x_1) \neq 0$)

<u>EXAMPLE</u> : $f(x) = x^3 - 2x - 5$

$x_1 = 2$

$f(x_1) = f(2) = -1$

$f'(x) = 3x^2 - 2$

$f'(x_1) = f'(2) = 10$

$$\Rightarrow \quad x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{-1}{10} = 2.1$$

3. REPEAT STEP #2, BUT BEGINNING WITH x_2 RATHER THAN x_1. THUS, THE THIRD APPROXIMATION IS

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

(PROVIDED $f'(x_2) \neq 0$)

<u>EXAMPLE</u>: $f(x) = x^3 - 2x - 5$

$f'(x) = 3x^2 - 2$

$x_1 = 2 \, , \; x_2 = 2.1$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.1 - \frac{f(2.1)}{f'(2.1)}$$

$$= 2.1 - \frac{(2.1)^3 - 2(2.1) - 5}{3(2.1)^2 - 2} = \cdots$$

$$= 2.0946$$

4. CONTINUE THE PROCESS

<u>ALGORITHM</u>:

x_1 = INITIAL GUESS

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$\vdots$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\vdots$$

EXAMPLE : $\cos x - x = 0$

$$f(x) = \cos x - x$$
$$f'(x) = -\sin x - 1$$

NOTE : $f(0) = \cos 0 - 0 = 1 > 0$

$f(1) = \cos 1 - 1 < 0$

SO THERE IS A SOLUTION BETWEEN 0 AND 1.

TAKE

$$x_1 = 0$$

THEN

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0 - \frac{f(0)}{f'(0)} = -\frac{1}{-1} = 1$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1 - \frac{\cos 1 - 1}{-\sin 1 - 1}$$

$$= 1 + \frac{\cos 1 - 1}{\sin 1 + 1}$$

$$= 0.75036\,3868 \quad \text{(FROM A CALCULATOR)}$$

AND CONTINUE AS LONG AS YOU PLEASE.

A WORD OF CAUTION : THINGS CAN GO SERIOUSLY WRONG :