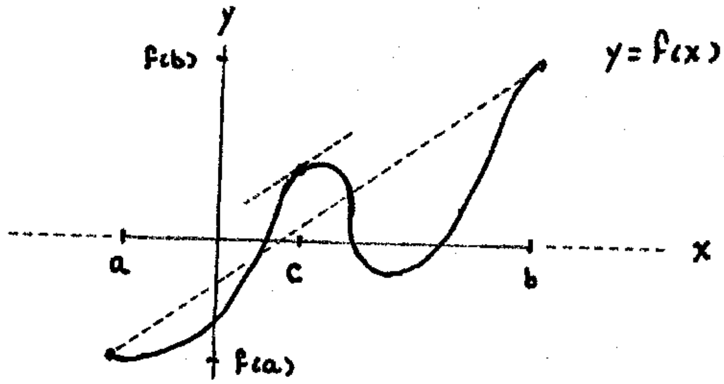
MEAN VALUE THEOREM

THE LAST TOPIC WE NEED TO COVER IS A THEOREM THAT SAYS SOMETHING FAIRLY "OBVIOUS" AND MAY SEEM A LITTLE TECHNICAL, BUT HAS CONSEQUENCES THAT ARE CRUCIAL FOR GETTING STARTED IN CALCULUS II NEXT TERM. HERE'S A PICTURE OF IT :



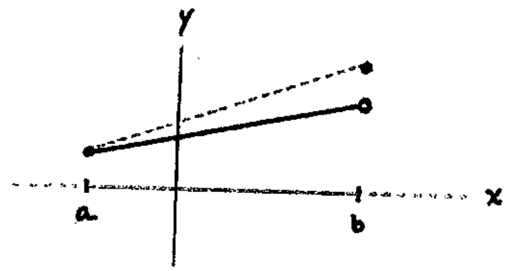
MEAN VALUE THEOREM : SUPPOSE $f(x)$ IS CONTINUOUS ON $[a,b]$ AND DIFFERENTIABLE ON (a,b). THEN THERE EXISTS A c IN (a,b) AT WHICH THE TANGENT LINE IS PARALLEL TO THE SECANT LINE JOINING THE POINTS $(a, f(a))$ AND $(b, f(b))$, I.E., AT WHICH

$$f'(c) = \frac{f(b)-f(a)}{b-a} .$$

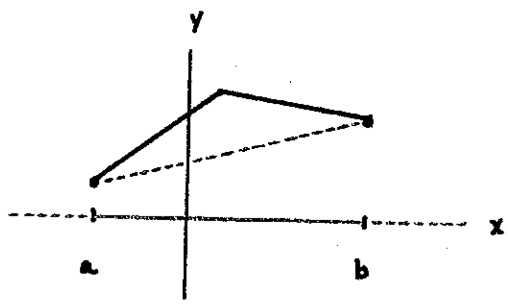
I WON'T PROVE THIS, BUT THERE ARE A NUMBER OF THINGS WE SHOULD NOTICE ABOUT IT .

1. THERE COULD BE MANY SUCH cs .

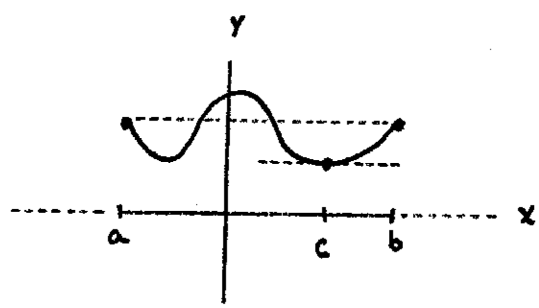
2. "CONTINUOUS ON $[a,b]$" IS NECESSARY, E.G.,



3. "DIFFERENTIABLE ON (a,b)" IS NECESSARY, E.G.,



4. IF $f(a) = f(b)$, THEN THE THEOREM SAYS THAT THERE EXISTS A C IN (a,b) AT WHICH $f'(c) = 0$.



THIS SPECIAL CASE OF THE MEAN VALUE THEOREM IS CALLED <u>ROLLE'S</u> <u>THEOREM</u>.

WE'LL VERIFY THE HYPOTHESES AND CONCLUSION OF THE MEAN VALUE THEOREM IN ONE EXAMPLE.

<u>EXAMPLE</u>: CONSIDER $f(x) = \sqrt{x-1}$ ON $[2,5]$.

$f(x)$ IS CONTINUOUS WHEN $x-1 \geq 0$, I.E., WHEN $x \geq 1$.
IN PARTICULAR, $f(x)$ IS CONTINUOUS ON $[2,5]$.

$f'(x) = \dfrac{1}{2\sqrt{x-1}}$ SO $f(x)$ IS DIFFERENTIABLE WHEN $x > 1$.
IN PARTICULAR, $f(x)$ IS DIFFERENTIABLE ON $(2,5)$.

$$\frac{f(b)-f(a)}{b-a} = \frac{f(5)-f(2)}{5-2} = \frac{\sqrt{5-1} - \sqrt{2-1}}{3} = \frac{1}{3}$$

THE MEAN VALUE THEOREM ASSERTS THAT, FOR SOME c IN $(2,5)$, $f'(c) = \frac{1}{3}$. LET'S FIND IT:

$$f'(x) = \frac{1}{3}$$

$$\frac{1}{2\sqrt{x-1}} = \frac{1}{3}$$

$$2\sqrt{x-1} = 3$$

$$4(x-1) = 9$$

$$x-1 = \frac{9}{4}$$

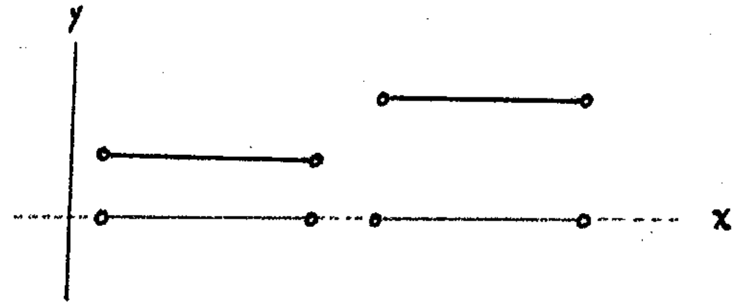
$$x = \frac{13}{4}$$

NOTICE THAT $\frac{13}{4}$ IS IN $(2,5)$ SO WE MAY TAKE $c = \frac{13}{4}$.

NOW WE DERIVE THE CONSEQUENCES OF THE MEAN VALUE THEOREM THAT WILL BE NEEDED IN CALCULUS II:

THE DERIVATIVE OF A CONSTANT FUNCTION IS ZERO. IT IS NOT TRUE, HOWEVER, THAT IF THE DERIVATIVE OF A FUNCTION IS ZERO EVERYWHERE ON ITS DOMAIN, THEN THE FUNCTION MUST BE CONSTANT.

E.G.,



HOWEVER, IF THE DOMAIN IS AN INTERVAL, THINGS ARE DIFFERENT.

THEOREM: IF $f(x)$ IS CONTINUOUS ON $[a,b]$ AND DIFFERENTIABLE ON (a,b) AND IF $f'(x)=0$ FOR EVERY x IN (a,b), THEN $f(x)$ IS A CONSTANT FUNCTION ON $[a,b]$.

HERE'S WHY: LET x BE ANY POINT IN $(a,b]$. THEN f IS CONTINUOUS ON $[a,x]$ AND DIFFERENTIABLE ON (a,x) SO THE MEAN VALUE THEOREM SAYS THAT, FOR SOME c IN (a,x)

$$\frac{f(x)-f(a)}{x-a} = f'(c)$$

BUT $f'(c)=0$ SO

$$f(x)-f(a)=0$$

I.E.,

$$f(x) = f(a).$$

f TAKES THE SAME VALUE AT EVERY x THAT IT TAKES AT a. IT'S CONSTANT!

<u>THEOREM</u> : IF $f(x)$ AND $g(x)$ ARE CONTINUOUS ON $[a,b]$, DIFFERENTIABLE ON (a,b) AND $f'(x) = g'(x)$ FOR EVERY x IN (a,b), THEN

$$g(x) = f(x) + C$$

FOR SOME CONSTANT C.

THIS IS EASY TO SEE : $g(x) - f(x)$ IS CONTINUOUS ON $[a,b]$, DIFFERENTIABLE ON (a,b) AND $(g(x) - f(x))' = g'(x) - f'(x) = 0$ ON (a,b). BY THE LAST THEOREM,

$$\begin{aligned} g(x) - f(x) &= C \\ g(x) &= f(x) + C \end{aligned}$$

FOR SOME CONSTANT C.

<u>EXAMPLE</u> : $f(x) = \arcsin x$ AND $g(x) = -\arccos x$ ARE CONTINUOUS ON $[-1,1]$, DIFFERENTIABLE ON $(-1,1)$ AND

$$f'(x) = \frac{1}{\sqrt{1-x^2}} = g'(x)$$

ON $(-1,1)$. THUS,

$$-\arccos x = \arcsin x + C$$

FOR SOME C. PLUG IN $x = 0$ TO GET

$$-\frac{\pi}{2} = 0 + C$$

SO $-\arccos x = \arcsin x - \frac{\pi}{2}$, I.E.,

$$\arccos x + \arcsin x = \frac{\pi}{2} .$$