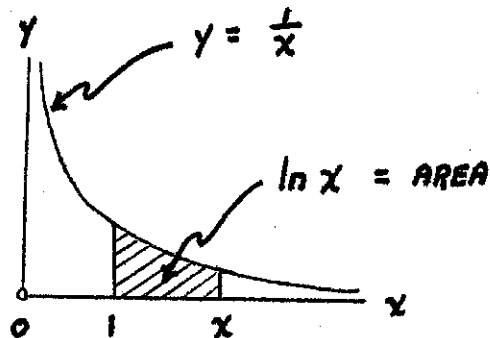


DERIVATIVES OF LOGARITHMIC, EXPONENTIAL AND INVERSE TRIG FUNCTIONS

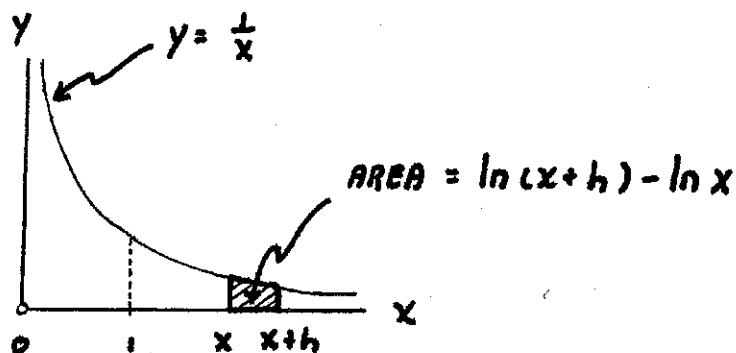
RECALL THE "CORRECT" DEFINITION OF $\ln x$ ($x > 0$):



NOW WE WANT TO FIND THE DERIVATIVE OF THE NATURAL LOGARITHM FUNCTION.

$$(\ln x)' = \lim_{h \rightarrow 0} \frac{\ln(x+h) - \ln x}{h}$$

HERE'S A PICTURE OF $\ln(x+h) - \ln x$:



THUS,

$$h \left(\frac{1}{x+h} \right) \leq \ln(x+h) - \ln x \leq h \left(\frac{1}{x} \right)$$

DIVIDE EVERYTHING BY h TO GET

$$\frac{1}{x+h} \leq \frac{\ln(x+h) - \ln x}{h} \leq \frac{1}{x}$$

SINCE $\lim_{h \rightarrow 0} \frac{1}{x+h} = \frac{1}{x}$ AND $\lim_{h \rightarrow 0} \frac{1}{x} = \frac{1}{x}$, THE SQUEEZE

THEOREM IMPLIES THAT

$$\lim_{h \rightarrow 0} \frac{\ln(x+h) - \ln x}{h} = \frac{1}{x}$$

AS WELL, THUS, FOR $x > 0$,

$$\boxed{(\ln x)' = \frac{1}{x}}$$

COMBINED WITH THE CHAIN RULE THIS GIVES THE USUAL THING:

$$\boxed{(\ln u(x))' = \frac{1}{u(x)} u'(x)}$$

FOR ANY POSITIVE FUNCTION $u(x)$.

$$\text{E.G., } (\ln \sqrt{x})' = \frac{1}{\sqrt{x}} (\sqrt{x})' = \frac{1}{\sqrt{x}} \left(\frac{1}{2\sqrt{x}} \right) = \frac{1}{2x}$$

OR, EVEN BETTER,

$$(\ln \sqrt{x})' = (\ln x^{\frac{1}{2}})' = \left(\frac{1}{2} \ln x \right)' = \frac{1}{2} \frac{1}{x} = \frac{1}{2x}$$

$$(\ln x)' = \frac{1}{x} \quad \text{FOR } x > 0$$

$$(\ln u(x))' = \frac{1}{u(x)} u'(x) \quad \text{FOR } u(x) > 0$$

FOR $x < 0$, $-x > 0$ SO $\ln(-x)$ IS DEFINED AND

$$(\ln(-x))' = \frac{1}{-x} (-x)' = \frac{1}{-x} (-1) = \frac{1}{x}$$

THUS,

$$(\ln x)' = \frac{1}{x} \quad \text{FOR } x > 0$$

$$(\ln(-x))' = \frac{1}{x} \quad \text{FOR } x < 0$$

SO

$$\boxed{(\ln |x|)' = \frac{1}{x} \quad \text{FOR } x \neq 0}$$

WITH THE CHAIN RULE,

$$\boxed{(\ln |u(x)|)' = \frac{1}{u(x)} u'(x)}$$

$$\text{E.G., } (\ln |\sin x|)' = \frac{1}{\sin x} \cos x = \cot x$$

LOGARITHMS WITH OTHER BASES: $\log_b x = \frac{\ln x}{\ln b}$ SO

$$\boxed{(\log_b x)' = \frac{1}{\ln b} \frac{1}{x}}$$

$$\text{E.G., } (\log_2 x)' = \frac{1}{\ln 2} \frac{1}{x}$$

WITH THE CHAIN RULE,

$$(\log_b u(x))' = \frac{1}{\ln b} \frac{1}{u(x)} u'(x)$$

WITH THE ABSOLUTE VALUES,

$$(\log_b |u(x)|)' = \frac{1}{\ln b} \frac{1}{u(x)} u'(x)$$

E.G., $(\log_3 |\tan x|)' = \frac{1}{\ln 3} \frac{1}{\tan x} (\sec^2 x)$

EXPONENTIAL FUNCTIONS :

$$y = e^x$$

$$\ln y = x$$

DIFFERENTIATE IMPLICITLY WITH RESPECT TO x :

$$(\ln y)' = (x)'$$

$$\frac{1}{y} y' = 1$$

$$y' = y$$

$$y' = e^x$$

$$(e^x)' = e^x$$

WITH THE CHAIN RULE

$$(e^{u(x)})' = e^{u(x)} u'(x)$$

$$\text{E.G., } (e^{\csc x})' = e^{\csc x} (-\csc x \cot x)$$

EXPONENTIAL FUNCTIONS WITH OTHER BASES :

$$b^x = e^{x \ln b}$$

SO

$$\begin{aligned} (b^x)' &= e^{x \ln b} (x \ln b)' \\ &= b^x \ln b \end{aligned}$$

$$\boxed{(b^x)' = b^x \ln b}$$

WITH THE CHAIN RULE,

$$\boxed{(b^{u(x)})' = b^{u(x)} \ln b u'(x)}$$

E.G.,

$$(5^{x^2})' = 5^{x^2} \ln 5 (2x)$$

EXAMPLES :

1. COMPUTE y' IF $y = \sin(e^x)$

$$\begin{aligned} y' &= \cos(e^x) (e^x)' = \cos(e^x) e^x \\ &= e^x \cos(e^x) \end{aligned}$$

2. COMPUTE y' IF $y = \exp \sqrt{1+5x^3}$

$$\begin{aligned} y' &= (e^{\sqrt{1+5x^3}})' = e^{\sqrt{1+5x^3}} (\sqrt{1+5x^3})' \\ &= e^{\sqrt{1+5x^3}} \left(\frac{1}{2} (1+5x^3)^{-\frac{1}{2}} (1+5x^3)' \right) \\ &= e^{\sqrt{1+5x^3}} \left(\frac{15x^2}{2\sqrt{1+5x^3}} \right) \\ &= \frac{15x^2 \exp \sqrt{1+5x^3}}{2\sqrt{1+5x^3}} \end{aligned}$$

3. COMPUTE y' IF $y = \ln \left(\frac{x^2 \sqrt[3]{7x-14}}{(1+x^2)^4} \right)$

SIMPLIFY FIRST !

$$\begin{aligned} y &= \ln (x^2 \sqrt[3]{7x-14}) - \ln (1+x^2)^4 \\ &= \ln (x^2) + \ln (7x-14)^{\frac{1}{3}} - 4 \ln (1+x^2) \\ &= 2 \ln x + \frac{1}{3} \ln (7x-14) - 4 \ln (1+x^2) \end{aligned}$$

$$\begin{aligned} y' &= 2 \left(\frac{1}{x} \right) + \frac{1}{3} \frac{1}{7x-14} (7x-14)' - 4 \frac{1}{1+x^2} (1+x^2)' \\ &= \frac{2}{x} + \frac{7}{3(7x-14)} - \frac{8x}{1+x^2} \end{aligned}$$

4. COMPUTE y' IF $y = \frac{x^2 \sqrt[3]{7x-14}}{(1+x^2)^4}$

LOGARITHMIC DIFFERENTIATION: TAKE \ln OF BOTH SIDES,
DIFFERENTIATE IMPLICITLY AND SOLVE FOR y' .

$$y = \frac{x^2 \sqrt[3]{7x-14}}{(1+x^2)^4} \Rightarrow \ln y = \ln \left(\frac{x^2 \sqrt[3]{7x-14}}{(1+x^2)^4} \right)$$

$$\ln y = 2 \ln x + \frac{1}{3} \ln(7x-14) - 4 \ln(1+x^2)$$

DIFFERENTIATE :

$$\frac{1}{y} y' = \frac{2}{x} + \frac{7}{3(7x-14)} - \frac{8x}{1+x^2}$$

$$y' = \left(\frac{2}{x} + \frac{7}{3(7x-14)} - \frac{8x}{1+x^2} \right) y$$

$$y' = \left(\frac{2}{x} + \frac{7}{3(7x-14)} - \frac{8x}{1+x^2} \right) \frac{x^2 \sqrt[3]{7x-14}}{(1+x^2)^4}$$

5. $y = x^{\sin x}$

$$\ln y = \ln (x^{\sin x})$$

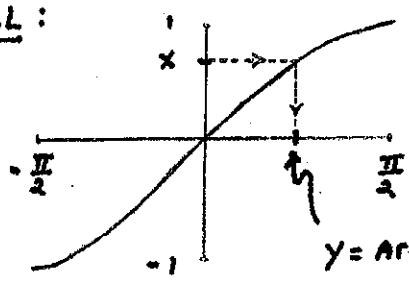
$$\ln y = (\sin x)(\ln x)$$

$$\frac{1}{y} y' = (\sin x) \left(\frac{1}{x} \right) + (\ln x)(\cos x)$$

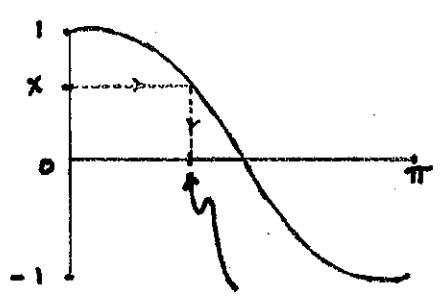
$$y' = \left(\frac{\sin x}{x} + (\ln x)(\cos x) \right) y$$

$$y' = \left(\frac{\sin x}{x} + (\ln x)(\cos x) \right) x^{\sin x}$$

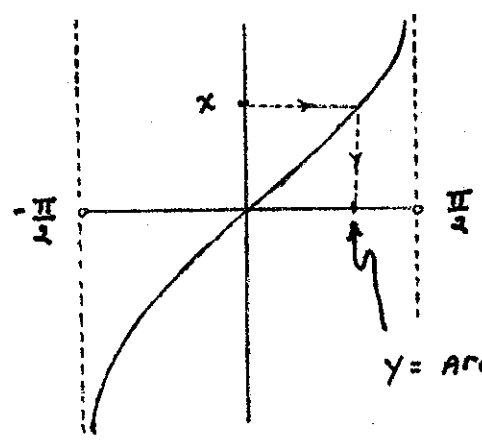
RECALL :



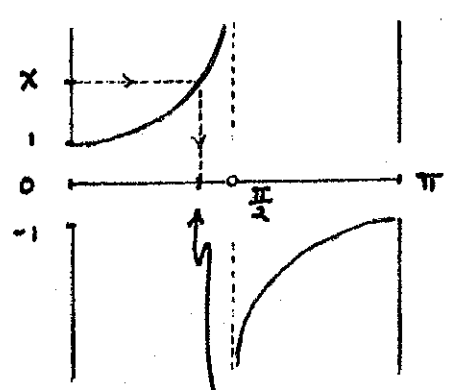
$y = \text{ARCSIN } X = \text{ANGLE IN } [-\frac{\pi}{2}, \frac{\pi}{2}] \text{ WHOSE SINE IS } X$



$y = \text{ARCCOS } X = \text{ANGLE IN } [0, \pi] \text{ WHOSE COSINE IS } X$



$y = \text{ARCTAN } X = \text{ANGLE IN } (-\frac{\pi}{2}, \frac{\pi}{2}) \text{ WHOSE TANGENT IS } X$



$y = \text{ARCSEC } X = \text{ANGLE IN } [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi] \text{ WHOSE SECANT IS } X$

DERIVATIVES :

$$y = \text{ARCSIN } X$$

$$\text{SIN } y = X$$

DIFFERENTIATE IMPLICITLY WITH RESPECT TO X :

$$(\text{SIN } y)' = (X)'$$

$$(\text{COS } y) y' = 1$$

$$y' = \frac{1}{\text{COS } y}$$

$$y' = \frac{1}{\sqrt{1 - \text{SIN}^2 y}}$$

$$\boxed{(\text{ARCSIN } X)' = \frac{1}{\sqrt{1 - X^2}}}$$

SIMILARLY,

$(\text{ARCCOS } X)' = -\frac{1}{\sqrt{1 - X^2}}$
$(\text{ARCTAN } X)' = \frac{1}{1 + X^2}$
$(\text{ARCSEC } X)' = \frac{1}{ X \sqrt{X^2 - 1}}$

COMBINED WITH THE CHAIN RULE WE GET THE USUAL THING :

$(\text{ARCSIN } u(x))' = \frac{1}{\sqrt{1-(u(x))^2}} u'(x)$
$(\text{ARCCOS } u(x))' = -\frac{1}{\sqrt{1-(u(x))^2}} u'(x)$
$(\text{ARCTAN } u(x))' = \frac{1}{1+(u(x))^2} u'(x)$
$(\text{ARCSEC } u(x))' = \frac{1}{ u(x) \sqrt{(u(x))^2-1}} u'(x)$

EXAMPLES :

$$1. (\text{ARCSIN } (x^3))' = \frac{1}{\sqrt{1-(x^3)^2}} (x^3)' = \frac{3x^2}{\sqrt{1-x^6}}$$

$$2. (\text{ARCSEC } (e^x))' = \frac{1}{|e^x|\sqrt{(e^x)^2-1}} (e^x)' = \frac{e^x}{e^x \sqrt{e^{2x}-1}} = \frac{1}{\sqrt{e^{2x}-1}}$$

$$3. f(x) = \ln(\text{ARCTAN } \sqrt{x}) \Rightarrow$$

$$\begin{aligned} f'(x) &= \frac{1}{\text{ARCTAN } \sqrt{x}} (\text{ARCTAN } \sqrt{x})' \\ &= \frac{1}{\text{ARCTAN } \sqrt{x}} \frac{1}{1+(\sqrt{x})^2} (\sqrt{x})' = \frac{1}{\text{ARCTAN } \sqrt{x}} \frac{1}{1+x} \frac{1}{2\sqrt{x}} \\ &= \frac{1}{2\sqrt{x}(1+x)\text{ARCTAN } \sqrt{x}} \end{aligned}$$

4. FIND ALL VALUES OF x AT WHICH $y = \text{ARCTAN } (x^2)$ HAS A HORIZONTAL TANGENT LINE.

$$y' = \frac{1}{1+(x^2)^2} (x^2)' = \frac{2x}{1+x^4}$$

$$y' = 0 \Rightarrow \frac{2x}{1+x^4} = 0 \Rightarrow 2x = 0 \Rightarrow x = 0$$

5. FIND y' BY IMPLICIT DIFFERENTIATION IF

$$\text{ARCSIN}(xy) = \text{ARCCOS}(x-y)$$

$$(\text{ARCSIN}(xy))' = (\text{ARCCOS}(x-y))'$$

$$\frac{1}{\sqrt{1-(xy)^2}} (xy)' = -\frac{1}{\sqrt{1-(x-y)^2}} (x-y)'$$

$$\frac{1}{\sqrt{1-x^2y^2}} (xy' + y) = -\frac{1}{\sqrt{1-(x-y)^2}} (1-y')$$

$$\frac{x}{\sqrt{1-x^2y^2}} y' + \frac{y}{\sqrt{1-x^2y^2}} = -\frac{1}{\sqrt{1-(x-y)^2}} + \frac{1}{\sqrt{1-(x-y)^2}} y'$$

MULTIPLY THROUGH BY $\sqrt{1-x^2y^2} \sqrt{1-(x-y)^2}$:

$$x \sqrt{1-(x-y)^2} y' + y \sqrt{1-(x-y)^2} = -\sqrt{1-x^2y^2} + \sqrt{1-x^2y^2} y'$$

$$(x \sqrt{1-(x-y)^2} - \sqrt{1-x^2y^2}) y' = -\sqrt{1-x^2y^2} - y \sqrt{1-(x-y)^2}$$

$$y' = -\frac{y \sqrt{1-(x-y)^2} + \sqrt{1-x^2y^2}}{x \sqrt{1-(x-y)^2} - \sqrt{1-x^2y^2}}$$

6. FIND $\frac{d^2y}{dx^2}$ IF $y = \text{ARCCOS}(3x)$

$$\frac{dy}{dx} = (\text{ARCCOS}(3x))' = -\frac{1}{\sqrt{1-(3x)^2}} (3x)' = -\frac{3}{\sqrt{1-9x^2}}$$

$$\frac{d^2y}{dx^2} = \left(-\frac{3}{\sqrt{1-9x^2}}\right)' = -3 \left((1-9x^2)^{-\frac{1}{2}}\right)'$$

$$= -3 \left(-\frac{1}{2} (1-9x^2)^{-\frac{3}{2}} (1-9x^2)'\right)$$

$$= \frac{3}{2} (1-9x^2)^{-\frac{3}{2}} (-18x) = -27x (1-9x^2)^{-3/2}$$