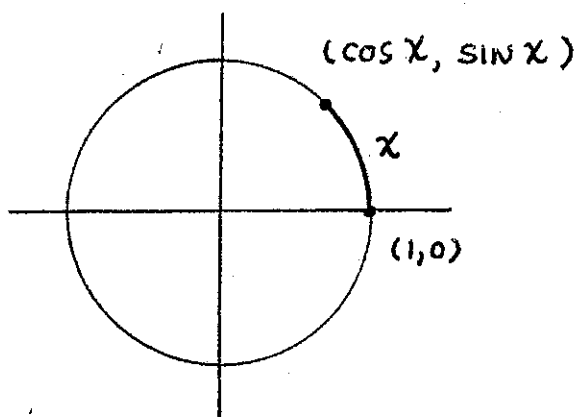


LIMITS OF TRIG FUNCTIONS : THE SQUEEZE THEOREM

$\cos x$ AND $\sin x$ ARE CONTINUOUS FOR ALL x

$\tan x$, $\cot x$, $\sec x$ AND $\csc x$ ARE CONTINUOUS FOR THOSE x AT WHICH THE DENOMINATOR IS NONZERO, E.G.,

$$\tan x = \frac{\sin x}{\cos x} \text{ IS CONTINUOUS FOR}$$

$$\text{ALL } x \text{ EXCEPT } \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}, \dots$$

I.E.,

$$\frac{(2n+1)\pi}{2}, \quad n = 0, \pm 1, \pm 2, \dots$$

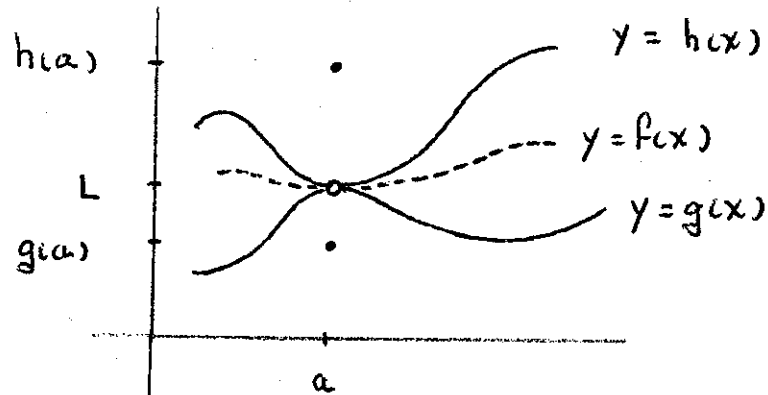
SOME LIMITS :

$$\lim_{x \rightarrow 0} \cos x = \cos 0 = 1$$

$$\lim_{x \rightarrow 0} \sin x = \sin 0 = 0$$

$$\begin{aligned} \lim_{x \rightarrow 1} \cos(x^2 - 1) &= \cos(\lim_{x \rightarrow 1} (x^2 - 1)) \\ &= \cos(1^2 - 1) = \cos 0 = 1 \end{aligned}$$

SOME LIMITS ARE NOT SO EASY AND REQUIRE A NEW TOOL :

THE SQUEEZE THEOREM :

$$\lim_{x \rightarrow a} g(x) = L$$

$$\lim_{x \rightarrow a} h(x) = L$$

$$g(x) \leq f(x) \leq h(x)$$

$$\Rightarrow \lim_{x \rightarrow a} f(x) = L \text{ ALSO}$$

EXAMPLE : $\lim_{x \rightarrow 0} x^2 \sin\left(\frac{1}{x^2}\right)$

NOTE THAT, FOR ANY $x \neq 0$,

$$-1 \leq \sin\left(\frac{1}{x^2}\right) \leq 1$$

SO, MULTIPLYING BY x^2 (WHICH IS POSITIVE),

$$-x^2 \leq x^2 \sin\left(\frac{1}{x^2}\right) \leq x^2$$

SINCE $\lim_{x \rightarrow 0} (-x^2) = 0$ AND $\lim_{x \rightarrow 0} (x^2) = 0$ WE CONCLUDE THAT

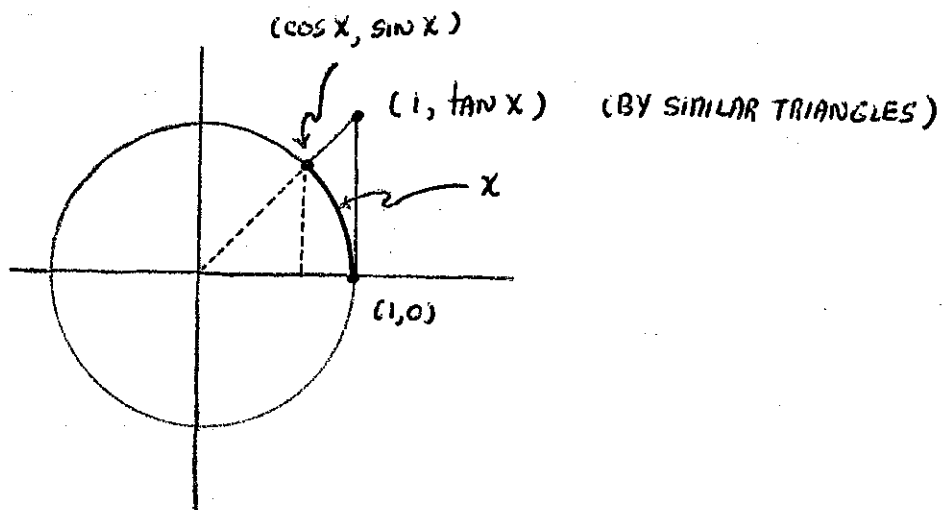
$$\lim_{x \rightarrow 0} x^2 \sin\left(\frac{1}{x^2}\right) = 0$$

ALSO.

A MUCH MORE IMPORTANT EXAMPLE : WE WILL SHOW THAT

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

REMEMBER
THIS ONE !



$$\text{AREA}(\triangle) \gg \text{AREA}(\text{sector}) \gg \text{AREA}(\triangle)$$

$$\frac{1}{2}(1)\tan x \gg \frac{1}{2}(1^2)x \gg \frac{1}{2}\cos x \sin x$$

$$\tan x \gg x \gg \cos x \sin x$$

$$\frac{1}{\cos x} \gg \frac{x}{\sin x} \gg \cos x$$

$$\cos x \leq \frac{\sin x}{x} \leq \frac{1}{\cos x}$$

SINCE $\lim_{x \rightarrow 0} \cos x = \cos 0 = 1$ AND $\lim_{x \rightarrow 0} \frac{1}{\cos x} = \frac{1}{\cos 0} = 1$,

THE SQUEEZE THEOREM IMPLIES THAT $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ ALSO,

SOME MORE LIMITS YOU CAN GET FROM THIS ONE :

1.

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0$$

REMEMBER
THIS ONE !

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} &= \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} \frac{1 + \cos x}{1 + \cos x} = \lim_{x \rightarrow 0} \frac{1 - \cos^2 x}{x(1 + \cos x)} \\ &= \lim_{x \rightarrow 0} \frac{\sin^2 x}{x(1 + \cos x)} = \lim_{x \rightarrow 0} \frac{\sin x}{x} \frac{\sin x}{1 + \cos x} \\ &= (1)(0) = 0 \end{aligned}$$

$$\begin{aligned} 2. \quad \lim_{x \rightarrow 0} \frac{\tan x}{x} &= \lim_{x \rightarrow 0} \frac{\frac{\sin x}{\cos x}}{x} = \lim_{x \rightarrow 0} \frac{\sin x}{x} \frac{1}{\cos x} \\ &= (1)(1) = 1 \end{aligned}$$

$$3. \quad \lim_{x \rightarrow 0} \frac{\sin 3x}{x} = \lim_{x \rightarrow 0} 3 \left(\frac{\sin 3x}{3x} \right) = 3 \lim_{x \rightarrow 0} \frac{\sin 3x}{3x}$$

NOW LET $t = 3x$ AND NOTICE THAT

$x \rightarrow 0$ IF AND ONLY IF $t \rightarrow 0$

$$= 3 \lim_{t \rightarrow 0} \frac{\sin t}{t} = 3(1) = 3$$

$$4. \quad \lim_{\theta \rightarrow 0} \left(\frac{\theta + \sin \theta}{\theta} \right) = \lim_{\theta \rightarrow 0} \left(1 + \frac{\sin \theta}{\theta} \right) = 1 + 1 = 2$$

LOGARITHMIC AND EXPONENTIAL FUNCTIONS ARE ALSO CONTINUOUS ON THEIR DOMAINS, AS ARE THE INVERSE TRIGONOMETRIC FUNCTIONS.